The nature of inviscid vortex breakdown

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We show that vortex breakdown appears as a jump bifurcation due to structural instability of swirling flows when their solution locally fails to exist and the flow transits to another stable or metastable state. The flow pattern inside the steady separation zone and, consequently, the vortex breakdown features depend on the flow history. Therefore, we take into account the flow pattern inside the separation zone. The stagnation zone model (without velocity jump) excels the traditional analytic continuation method (leading to a recirculation zone) in that solutions always exist, and, for large enough inflow swirl, exhibit nonuniqueness and folds due to smooth variations of flow parameters, thus predicting the experimentally observed hysteretic jump transitions. © 1997 American Institute of Physics. [S1070-6631(97)00701-0]

Despite extensive investigations of vortex breakdown, well summarized in the literature,1 there is yet no consensus as to what constitutes vortex breakdown, let alone a clear explanation for it. Our approach2 to vortex breakdown is quite different from traditional approaches. We do not consider vortex breakdown as an abrupt transition in the physical space (as a shock wave or a hydraulic jump) between two flow states, related to wave phenomena. We consider vortex breakdown as a bifurcation (abrupt) jump of the entire flow to another state, due to smooth variations of flow parameters; nonuniqueness associated with such a jump (say, a fold) is a necessary feature of vortex breakdown. Here, we address the following question: why and how does this jump occur?

The steady Euler equations for an inviscid, incompressible, axisymmetric flow in cylindrical coordinates \((r, \varphi, z)\) can be expressed in terms of the Stokes streamfunction \(\psi\) as

\[
\Delta \psi = r \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial \psi}{\partial r} \right] + \frac{\partial^2 \psi}{\partial z^2} = r^2 H' - \Gamma' = G(r, \psi);
\]

\[
v_z = (1/r)(\partial \psi / \partial r), \quad v_r = -(1/r)(\partial \psi / \partial z),
\]

where \(\Gamma(\psi)\) and \(H(\psi)\) are circulation and Bernoulli head, and \(v_r, v_z\) are the velocity components; here the prime denotes differentiation with respect to \(\psi\).

The functions \(H(\psi)\) and \(\Gamma(\psi)\) must be determined from boundary and initial conditions. A regular solution3 is an exception because only the boundary conditions need to be specified. In general, the solution may not be regular; moreover, in the presence of sufficiently large inflow vorticity, separation is inevitable, and the functions \(\Gamma(\psi)\) and \(H(\psi)\) are undetermined inside the steady separation zone. Traditionally,1 the indeterminacy is overcome by imposing additional conditions, such as using analytic continuation or a stagnation zone model.

If \(H(\psi)\) and \(\Gamma(\psi)\) are given for all \(\psi\), existence and uniqueness of the solution of (1) is not guaranteed. But if the function \(G(r, \psi)\) in (1) is bounded for all \(\psi\), then (1) has a solution (Ref. 4, p. 369). If, in addition, \(G\) is a monotonically increasing function of \(\psi\), then the solution is unique (Ref. 4, p. 372). Clearly, there are three basic cases.

(a) Singular: when \(G\) is an unbounded (for example, linear) function of \(\psi\). Solutions may not exist. However, if a steady flow is a time limit of an unsteady problem (the only physically acceptable situation), the solution of (1) must exist due to the existence and uniqueness theorems for unsteady inviscid flows. Hence, \(H(\psi)\) and \(\Gamma(\psi)\) can not be chosen arbitrarily.

(b) Semiregular: when \(G\) is bounded, but is a decreasing or a nonmonotonic function of \(\psi\). The solution of (1) exists but, in general, is not unique.

(c) Regular: when \(G\) is bounded and an increasing function of \(\psi\). A unique solution always exists.

As shown later, only case (b) involves vortex breakdown.

Consider a semi-infinite \((0 \leq z \leq \infty)\) pipe of radius \(R\) (Fig. 1) with inflow conditions:

\[
v_z = u_0 = \text{const},
\]

\[
\omega_z = \begin{cases} 2\Omega, & 0 \leq r \leq r_0, \\ 0, & r_0 < r \leq R, \end{cases}
\]

\[
\omega_\varphi = \begin{cases} -\alpha u_0 r^2, & 0 \leq r \leq r_0, \\ 0, & r_0 < r \leq R. \end{cases}
\]

(2)

We nondimensionalize all quantities by the uniform inflow velocity \(u_0\) and the inflow core radius \(r_0\). The boundary conditions are \(\psi|_{r=0} = 0, \psi|_{r=R} = r^2/2; \psi|_{z=0} = r^2/2; \psi|_{z=\infty}\) is bounded, and \(\partial \psi / \partial n\) is continuous on the interface \(\psi = \frac{r^2}{2}\), case dividing the vortical and irrotational regions. Using (2), we find

\[
G(r, \psi) = \begin{cases} (k^2/2 + \alpha) r^2 - k^2 \psi, & \psi \leq \frac{r^2}{2}, \\ 0, & \psi > \frac{r^2}{2}. \end{cases}
\]

(3)

Here nondimensional \(k = 2\Omega r_0 / u_0\) and \(\alpha\) are measures at the inflow of axial \(\omega_z\) (or swirl) and azimuthal \(\omega_\varphi\) (i.e., cyclostrophic imbalance) vorticities, respectively. Thus, there are three control parameters: \(k, \alpha\), and \(R\). We have to find the nondimensional sizes \(r_1\) and \(r_2\) (see Fig. 1) and then all other

FIG. 1. Flow pattern for a Rankine vortex with axial flow and azimuthal vorticity inflow. Here \(v_z\) and \(v_\varphi\) are the profiles of the axial and azimuthal velocity components; \(R\) is the radius of the pipe; \(r_0\) is the radius of the vortex core at the entrance; \(r_1\) is the radius of the vortex core downstream; \(r_2\) is the radius of the separation zone.
flow characteristics. The use of (3) for all $\psi$ implies analytic continuation into the region $\psi<0$. In this case $G$ is unbounded, and it is a decreasing function of $\psi$ for $\psi<\frac{3}{4}$. Therefore, according to our classification, (3) is singular [i.e., case (a)] and we can expect nonexistence, nonuniqueness, or other unusual solution features.

We seek a solution of the type $\psi(r,z) = \psi_0(r) + \psi_1(r,z)$, where $\psi_0$ is a $z$-independent solution. If we assume that there is a transition in the pipe to a flow at $z=\infty$ consisting of a cylindrical central vortical region of radius $r_1$ (Fig. 1) and a cylindrical outer irrotational domain, then $\psi_0(r)$ is the solution as $z\to\infty$. Since the same stream surface separates rotational and irrotational flows, $\psi_0(r_1) = \frac{3}{4}$; $\psi_0(0) = 0$. The corresponding solution is

$$\psi_0 = \left( \frac{3}{4} + \frac{\alpha}{k^2} \right) y^2 - Ar J_1(kr),$$

where

$$A = \left( \frac{3}{4} r_1^2 - 1 \right) + \alpha r_1^2 k J_1(kr_1)/[r_1 J_1(kr_1)].$$

Using (4) and (1), we find that the axial velocity in the flow core is given by

$$v_\perp = 1 + 2\alpha k^2 - Ak J_0(kr), \quad r < r_1.$$  \(5\)

On the other hand, $v_\perp(r)$ is constant in the outer irrotational zone:

$$v_\perp = (R^2 - 1)/(R^2 - r_1^2), \quad r > r_1.$$  \(6\)

Equating (5) and (6), we obtain the matching equation for analytic continuation,

$$\frac{\alpha(2y - k_1 J_0)}{k_1^2} = \frac{1 - x}{2} k_1 J_0 - \frac{1 - x}{y} J_1,$$

where $y = 1/R^2$ is the relative area of the inflow vortex core, $x = 1/r_1^2$ is the far downstream core variation parameter ($x<1$ corresponds to core expansion), $J = J(k_1)$, $k_1 = kr_1$, and $k = k_1 \sqrt{x}$. The axial velocity on the axis $V = v_\perp(0)$ is then

$$V = 1 - \frac{k_1(1 - x)}{2y} - \frac{\alpha}{2} k_1 J_1.$$  \(7\)

When $\alpha = 0$, (7) has two solutions: the trivial solution $(x=1$ and $V=1)$, and a nontrivial solution satisfying

$$- k_1 J_0 / y = 2y J_1.$$  \(8\)

Since $x>y$, the nontrivial solution may be physical only if $\mu_0 < k_1 < \mu_1$, where $\mu_0 = 2.4$ and $\mu_1 = 3.83$ are the first roots of the Bessel functions $J_0$ and $J_1$, respectively. Results for a rather slender vortex ($y = 0.1$) and various values of $\alpha$ are shown in Fig. 2. For $\alpha = 0$ the nontrivial solution along with the trivial solution $V=1$ represent the transcritical bifurcation diagram. At the bifurcation point B, both solutions coincide. According to bifurcation theory, if the solution branch AB is stable, the branch BD becomes unstable, and vice versa. We assume that AB is stable, consistent with analysis.6 Hence, we expect that a flow with $\alpha = 0$ and swirl number $k < k^*$ (including the swirl-free case $k = 0$) remains unchanged along the entire pipe length. Here, $k^*$ satisfies the equation $k^* J_0(k^*) = 2y(1 - y)$ and hence depends on $y$; for $y = 0$, $k^* = \mu_0 = 2.4$. However, for $k > k^*$ we should expect a flow transition to another stable state BC with $x>1$, i.e., to a decreasing vortex core size. This is a new phenomenon, vortex contraction. Once again, since BC is stable, branch OB is unstable. The variation $V(k)$ for $\alpha = 0$, shown in Fig. 2, reveals an unexpected feature: contrary to observations, a reversed flow appears when $k$ is small enough, but $V$ becomes positive as $k$ (i.e., swirl) increases. Although also predicted by wave theory for an infinitely long pipe this result indicates that branch OB is unstable and physically unrealizable. Note that $\alpha = 0$ is inherent to wave theory.

Thus, analytic continuation in a semi-infinite pipe with cyclotrophic balance ($\alpha = 0$) exhibits no vortex breakdown. However, since the transcritical bifurcation is structurally unstable,10 perturbations in the form of an infinitesimal cyclotrophic imbalance (i.e., small $\alpha > 0$) enable a jump bifurcation with the appearance of folds. Another notable feature is that $V(k) \to -\infty$ at some finite inflow swirl $k = k_{\infty}$, and the curve $CB$ ($\alpha = 0$) serves as an asymptote for the whole family of solutions. The global nonexistence of solutions (for a fixed $\alpha$) occurs when $k$ exceeds $k_{\infty}$. If $k < k_{\infty}$, we have local loss of solution, as in case $\alpha = 1$ in Fig. 2. For positive $V$, global nonexistence is observed for right limit points of curves $\alpha = 0.25$ and 0.5 in Fig. 2 (as no other solution exists at the same $k$). For $\alpha = 1$, the lower fold (which exists only for $k < k_{\infty}$) could correspond to a new stable branch of solutions) arises only for $V < 0$, i.e., inside the recirculation zone, while a stagnation point appears with increasing $k$ gradually without a jump. In our terminology, this is a case of internal separation, not vortex breakdown. These unexpected and unphysical features show that analytic continuation is inappropriate.

According to (3), case $k = 0$ is semiregular because $G(r,\psi)$ is a bounded decreasing function of $\psi$ so that a solution exists for any $\alpha$. At $k = 0$ and sufficiently large $\alpha$, we observe "vorticity breakdown" having features similar to vortex breakdown: the appearance of a stagnation point, strong core expansion, folds in parameter space, and hysteretic transitions.

Now we consider the steady flow model with a stagnation zone that results when the flow evolution starts from rest or as an irrotational motion. The stagnation model is different from (3) by the requirement that $G = 0$ for $\psi < 0$. The boundary conditions—namely $\psi(\bar{r}^2) = 0, v_\perp(\bar{r}^2) = 0, \psi(\bar{r}_1)$
FIG. 3. Bifurcation diagram for a stagnation zone flow model in case of slender vortex \( (\gamma=0.01) \) with inflow cyclostrophic balance \((\alpha=0); r_2(k) \) is the stagnation zone radius, \( x(k) \) the relative area of the vortex core far downstream, B is a transcritical bifurcation point; BLMEB is a hysteric loop.

\[ r_1 = \frac{1}{2} \text{ and } v_s(r_1) \text{ satisfies (6)} \] —enable the determination of \( r_1 \) and stagnation zone radius \( r_2 \).

An example for a slender vortex with \( \gamma=0.01 \) and \( \alpha=0 \) is depicted in Fig. 3. Curve \( r_2(k) \) demonstrates a subcritical bifurcation at point P. The left limit points of \( r_2(k) \) and \( x(k) \) are on the same vertical line ME. Thus, contrary to analytic continuation (Fig. 2), the stagnation zone solution \((V=0)\) corresponding to a stable lower branch \( x(k) \) exists to the right of M. One can observe a hysteresis phenomenon with loop BLMEB (for \( V \)). The states between M and L (or E and B) are metastable; i.e., transition between a trivial regime (EB) and the stagnation zone regime (ML) can occur anywhere in the intervals EB and ML, but not outside these intervals. The lower arrows correspond to the “latest” transitions; the upper arrows correspond to “earliest” ones. We see that increasing \( k \) can cause an abrupt jump (at point \( P \)) to the stagnation zone of the finite size \( r_2 = QM \), and, consequently, the stagnation point inside the pipe abruptly arises at a finite distance from the inflow. Such an appearance of the final size is obviously related to subcritical bifurcation with fold. For \( \alpha=0 \), comparison of Fig. 2 with Fig. 3 shows that regularization of case (a) by allowing the possibility of stagnation zone immediately leads to vortex breakdown due to the left fold in Fig. 3.

The smooth curve \( x(k) \) for the perturbed transcritical bifurcation \( B \) \((\gamma=0.1, \text{ and } \alpha=0.1) \) is shown in Fig. 4, where \( x_r \) and \( x_s \) are branches for recirculation and stagnation (which exists to the right of point P) zone cases, respectively; \( V = v_s(0) \) is the centerline velocity. We see that in the presence of an upstream swirl, folds, vortex breakdown, and vortex contraction arise along with nonuniqueness, but the nonexistence (in case of recirculation) disappears and a solution exists for any \( k \) in the stagnation zone model. This implies the surprising result that the onset of vortex breakdown depends on the flow pattern (i.e., on the flow history) inside the separation zone. For a swirl-free flow, an increase in \( \alpha \) leads only to supercritical bifurcations (not shown), when stagnation zones arise gradually without any jump transition; thus, the stagnation zone model predicts internal separation for swirl-free flows without any vorticity breakdown.

We conclude as follows. A necessary condition for vortex breakdown (contrary to the wave theory) is the departure (even weak) from the unperturbed structurally unstable state. The departure can occur for several reasons, such as weak flow divergence,\(^3,7\) viscosity,\(^8\) finite pipe length,\(^6\) inflow azimuthal vorticity or admissibility of a stagnation zone (i.e., boundedness of \( G \)). However, contrary to Refs. 3, 7, and 8, where vortex breakdown also appears as a jump transition, in our approach the result of the transition depends on the flow pattern inside the separation zone (i.e., on initial conditions). Therefore, in our viewpoint vortex breakdown is not a threshold phenomenon: it may occur within some range of the swirl, and the actual flow behavior depends on the flow history.

Thus, we argue that the occurrence of a stagnation zone is the most relevant model for vortex breakdown in inviscid flows. Besides hysteretic transitions, our theory also predicts “vortex contraction” at large swirl (see Fig. 4) due to a “blue-sky” bifurcation and also a “finite distance” effect: with increasing swirl, a stagnation zone of nonzero size abruptly appears far downstream, and, consequently, a stagnation point abruptly arises at a finite distance from the inflow.

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5M. Golubitsky and D. Schaeffer, Singularities and Groups in Bifurcation Theory (Springer Verlag, New York, 1985), p. 140.