Vortex sinks with axial flow: Solution and applications

Vladimir Shtern, Anatoly Borissov, and Fazle Hussain
Department of Mechanical Engineering, University of Houston, Houston, Texas 77204-4792

(Received 2 May 1996; accepted 9 June 1997)

In this paper we develop a new class of analytical solutions of the Navier–Stokes equations and suggest ways to predict and control complex swirling flows. We consider vortex sinks on curved axisymmetric surfaces with an axial flow and obtain a five-parameter solution family that describes a large variety of flow patterns and models fluid motion in a cylindrical can, whirlpools, tornadoes, and cosmic swirling jets. The singularity of these solutions on the flow axis is removed by matching them with swirling jets. The resulting composite solutions describe flows, consisting of up to seven separation regions (recirculatory “bubbles” and vortex rings), and model flows in the Ranque–Hilsch tube, in the meniscus of electrosprays, in vortex breakdown, and in an industrial vortex burner. The analytical solutions allow a clear understanding of how different control parameters affect the flow and guide selection of optimal parameter values for desired flow features. The approach permits extension to swirling flows with heat transfer and chemical reaction, and have the potential of being significantly useful for further detailed investigation by direct or large-eddy numerical simulations as well as laboratory experimentation. © 1997 American Institute of Physics. [S1070-6631(97)02110-7]

I. INTRODUCTION

Despite the long history of research in swirling flows, some problems crucial for applications remain unexplored, and many are unresolved. The primary difficulty is that swirling flows are typically complicated, consisting of a few recirculatory domains. Moreover, these flows have paradoxical features, including such striking phenomena as collapse (i.e., strong concentration of the axial and angular momenta), bistability (i.e., two stable states), the Ranque effect of thermal separation, and vortex breakdown. While collapse and bistability are observed in swirl-free flows as well, the Ranque effect and vortex breakdown are specific to swirling flows. Careful analyses are needed to predict and control these effects in natural and technological applications. Here we present a powerful new approach for such an analysis based on the generalization of the classical vortex-sink flow.

The planar vortex-sink (or vortex-source) flow is one of the simplest solutions of the Euler and Navier–Stokes equations. Such solutions (especially point vortices) are widely used as building blocks of more complex motions and as models of practical flows (see Ref. 4 for examples). Here, the planar vortex sink is generalized to cover axisymmetric vortex sinks with an axial flow. This is a new solution for a viscous incompressible fluid, incorporating an axial flow with a radial shear. The incorporation of the axial flow significantly enriches the family of solutions, increasing the number of dimensionless parameters for the velocity field up to five, and hence enabling the modeling of a larger variety of flows.

Prior analytical solutions related to ours dealt with swirl-free flows. A solution describing a flow near an infinite porous cylinder with uniform suction was found independently by Wuest,5 Lew,6 Yasuhara,7 and Stuart8 (Stuart also studied flow along a corner). Berman9 studied a similar flow, but in a porous annulus. Wang’s review10 and our own literature search revealed no other works on this subject. Unfortunately, the results5–9 address very specific flows of limited practical interest. In contrast, our solution addresses swirling flows of widespread technological importance.

The vortex-sink relation for radial and swirl velocities is a common feature of many swirling flows, due to (i) conservation of angular momentum in regions where viscous diffusion is negligible; and (ii) entrainment of ambient fluid by near-axis jet-like flows. The jet axis serves as a line sink for ambient fluid, as shown by Schlichting11 for swirl-free jets and by Long12 for swirling jets. The vortex sink is, in particular, an asymptotic solution for Long’s jet, as the distance from the axis becomes large; for other examples see Ref. 13.

The vortex-sink region is observed not only in open but also in confined swirling flows. Far from boundaries, a flow is typically oblivious of much of the constraint posed by boundary conditions. For instance, flows in vortex tubes13 and vortex generators14 are strongly asymmetric near tangential inlet nozzles, but become nearly axisymmetric, even at downstream distances comparable with the nozzle diameter. Also, the no-slip condition on sidewalls does not influence the main flow of interest, outside the boundary layer. For these reasons, the fine details of boundary conditions can be reasonably omitted for studies of robust features of swirling flows, as undertaken here.

Note that there are a few characteristics (governed by conservation laws) whose values are crucial for large-scale flow patterns, including the entrainment rate (here the radial Reynolds number Re) and the angular momentum (the swirl Reynolds number Ω). The reduction of detailed boundary conditions to a few integral characteristics is consistent with the universality of the vortex-sink region observed in a variety of swirling flows. Using just these characteristics—Re and Ω, for the radial and swirl velocity—we find that only three more parameters are required to specify the axial velocity in an axisymmetric flow. These parameters characterize contributions to the velocity profile from an outer free stream, due to an axial pressure gradient, and radial conver-
gence of the flow (as shown in Sec. II). Thus, the goal of this paper is to develop analytical solutions governing several important features common to a wide range of swirling flows. As a first step we obtain here a generalized vortex-sink solution.

The generalized vortex sink has a compact algebraic representation and a clear physical explanation. These features help to elucidate the intriguing mechanisms in the swirling flows listed above. In particular, the generalized vortex sink is applied here to model the strong momentum concentration in whirlpools, tornadoes, and cosmic jets. Also, the new solution explains the general mechanism of vortex filament formation.

A further extension of the generalized vortex sink is its use as an outer solution and the avoidance of its singular behavior on the axis by matching with inner solutions, such as swirling jets. The resulting composite solutions enable modeling of internal (i.e., away from walls) separation, which is a rather subtle but common phenomenon in swirling flows. The separation causes the appearance of recirculatory domains, which are either semi-infinite or compact (bubble- and torus-shaped) regions. The recirculatory domains are essential features of vortex breakdown, of the Ranque effect, and of vortex combustion.

The composite solutions describe complex flow patterns including up to seven recirculatory domains and model here vortex breakdown in sealed cylindrical cans, vortex burners, and Ranque–Hilsch tubes. Note that our analytical modeling covers the entire range of control parameters. This advantage enables classification of all possible flow regimes, as well as identification of the optimal parameter values for applications. Of course, for parameter values thus selected, more detailed flow features can be studied experimentally and by direct numerical simulation.

The remainder of this paper is organized as follows. The generalized vortex-sink solution is derived in Sec. II. A classification of the stream surface shapes is presented in Sec. III. Relevant inner solutions are discussed in Sec. IV and matched with the vortex sink in Sec. V, where a variety of flow patterns covered are shown. Examples of applications of the generalized vortex sink and composite solutions are given in Sec. VI and Sec. VII. Based on our new solution, we discuss a general mechanism of vortex filament formation in Sec. VIII. The key results are summarized in Sec. IX, where possible extensions are elaborated on as well.

II. GENERALIZED VORTEX SINK

We consider a steady swirling flow, where the velocity is independent of the axial coordinate \( z \). In this case, the Navier–Stokes equations for the radial velocity, \( v_r \), and swirl, \( v_\phi \), are decoupled from the equation for the axial velocity, \( v_z \). Here, \( v_r, v_\phi \), and \( v_z \) are the velocity components in cylindrical coordinates \( (r, \phi, z) \). For axisymmetric \( v_r \) and \( v_\phi \) (\( v_z \) may depend on \( \phi \)), the continuity equation reduces to \( d(v_r/v_r)dr=0 \) (i.e., \( v_r/\text{const} \)) corresponding to sink (or source) flow:

\[
v_r = Q(2\pi r)^{-1} = nr^{-1} \text{Re}.
\] (1a)

Here \( Q = 2\pi
r \text{Re} \) is the flow rate per unit axial length through a cylindrical surface, \( r = \text{const} \); \( \nu \) is the kinematic viscosity. The equation for swirl reduces to

\[
v_r r^{-1} d(r v_\phi)/dr = v_z (r^{-1} dr/dr (r dv_\phi/dr) - r^{-2} v_\phi).
\]

Multiplying by \( r^2 \nu \) and introducing \( \xi = \ln(r/\text{Re}) \), with \( r_0 \) as a length scale, yields

\[
\text{Re}(r \phi + v_\phi) = v_\phi - v_\phi,
\]

where the prime denotes differentiation with respect to \( \xi \). Note that the use of \( \xi \) results in constant coefficients. A general solution is

\[
v_\phi = C_1 \xi^{\text{Re}+1} + C_2 r^{-1}.
\] (1b)

The second term corresponds to the potential swirl, and the first term represents the solid-body rotation at \( \text{Re} = 0 \). In the context of this paper, we consider the particular solution \( C_1 = 0 \) [the role of the first term in (1b) is discussed in Sec. IV B]. In this case, \( v_\phi = K(2\pi r)^{-1} \), where \( K = 2\pi C_2 \) is the circulation along a circle with \( r = \text{const} \) and \( z = \text{const} \).

Throughout our analysis, we use the dimensionless parameter \( \Gamma = rv_\phi/v_f = K(2\pi r)^{-1} \), which is the swirl Reynolds number. The \( (v_r, v_\phi) \) flow is the well-known vortex sink (vortex source) for \( \text{Re} < 0 \) (\( \text{Re} > 0 \)).

The primary objective of our study is to generalize the vortex sink by adding an axial flow. Note that a \( z \)-independent axial flow does not affect the continuity equation and the momentum equations for \( v_r \) and \( v_\phi \). The governing equation for \( v_r \) reduces to the form

\[
\text{Re} W_{\xi} + \Gamma W_{\phi} = W_{\xi\xi} + W_{\phi\phi},
\] (1c)

where \( W = vr_{f0}/\nu \) and the subscripts denote differentiation. It is worth noting that one could also consider a nonaxisymmetric axial flow with an axisymmetric vortex sink given by (1a) and (1b).

Since the coefficients in (1c) are constant, the axial velocity has normal-mode solutions, \( W = \exp(a \phi + im \phi) \); the azimuthal wave number \( m = 0, \pm 1, \pm 2, ..., \) and \( a = a_1 \) or \( a_2 \) are the roots of the dispersion relation, \( a^2 - a \text{Re} m^2 - im \Gamma = 0 \). A general solution of (1c) is a superposition of normal modes, whose coefficients are determined by the boundary conditions. However, the applications of such nonaxisymmetric solutions are unknown to us, so that we focus here on the more practical axisymmetric case \( m = 0 \). The axisymmetric solution is \( W = W_c + W_s (r/\text{Re})^2 \), where \( W_c \) and \( W_s \) are constants.

A further generalization includes the inclusion of an axial pressure gradient, i.e., \( \partial p/\partial z = \text{const} \neq 0 \). For this case, a term \( P \exp(2\xi) \) must be added to the left-hand side of (1c), where \( P = P_0 (\rho v_f^2)^{-1} \partial p/\partial z \) is a dimensionless parameter characterizing the axial pressure gradient; \( \rho \) is the density. Then, the solution for \( W \) becomes \( W = W_c + W_s (r/\text{Re})^2 + W_p (r/\text{Re})^2 \), where \( W_p = P/(4 - 2 \text{Re}) \); the third term represents the contribution of the axial pressure gradient.

For flow in an unbounded domain \( 0 \leq r < \infty \), \( W_c \) is the velocity on the axis for \( \text{Re} > 0 \) or at infinity for \( \text{Re} < 0 \) and \( W_p = 0 \). In a moving Galilean frame, one can enforce \( W_c = 0 \). However, to model some flows (e.g., within vortex tubes), it is relevant to choose the frame such that zero axial
velocity occurs neither at \( r = 0 \) nor \( r = \infty \), but in-between (e.g., on the sidewall). Thus, it is useful to retain \( W_r \) as a free parameter characterizing the uniform part of the axial flow. The other parameters, \( W_\rho \) and \( W_z \), characterize the non-uniform shear of the axial velocity induced by the axial pressure gradient and the radial advection, respectively. Thus, the velocity field,
\[
v_r = \text{Re} \ n r, \tag{2a}
\]
\[
v_\theta = \Gamma n r, \tag{2b}
\]
\[
v_z = \left[ W_\rho + W_\rho (r/r_0)^2 + W_z (r/r_0) \text{Re} \right] n r / r_0, \tag{2c}
\]
is the new generalized vortex-sink solution, which satisfies the Navier–Stokes equations and includes five dimensionless parameters: \( \text{Re}, \Gamma, W_\rho, W_z, \) and \( W_\rho \). The vortex sink (2a), (2b), which is a known solution, is generalized here by an axial flow (2c). The axial flow includes a parabolic part [the first two terms in (2c)], which is similar to pipe flow and is independent of the vortex-sink component. In contrast, the last term in (2c) depends on the radial flow rate \( \text{Re} \) and is interpreted below as the entrainment effect of a near-axis jet.

Since the velocity in (2) depends on the radial coordinate only (through power-law functions), the streamline equation \( d\Gamma/r/s = v \) can also be explicitly integrated to yield
\[
z/r_0 = z_0/r_0 + a(r/r_0)^2 + b(r/r_0)^{\text{Re}+2} + c(r/r_0)^4, \tag{3a}
\]
\[
\phi = \phi_0 + S \ln(r/r_0), \tag{3b}
\]
where \( a = W_\rho (2 \text{Re})^{-1}, \quad b = W_\rho [\text{Re}(\text{Re}+2)]^{-1}, \quad c = W_\rho (4 \text{Re})^{-1}, \) and the swirl parameter \( S = \phi_\theta / v_r = \Gamma / \text{Re}. \)

The velocity field is identical on axisymmetric stream surfaces (3a), differing only by a shift \( z_0 \) along \( z \). Streamlines curve around on these surfaces, and their projection on \( z = \text{const} \) planes are logarithmic spirals, as is clear from (3b). The planar vortex sink is a particular case corresponding to \( a = b = c = 0 \) in (3).

To “visualize” flow patterns for comparison with experiments, it is convenient to use the Stokes stream-function \( \Psi; \) i.e., \( v_r = -r^{-1} \partial \Psi / \partial z \) and \( v_\theta = r^{-1} \partial \Psi / \partial r \). According to (2)–(3), the dimensionless streamfunction, \( \phi = \Psi(nr_0 \text{Re}^{-1}) \), has the form
\[
\phi = a(r/r_0)^2 + b(r/r_0)^{\text{Re}+2} + c(r/r_0)^4 - z/r_0. \tag{4}
\]

Finally, the pressure \( p \) for this solution family has the distribution
\[
p = p_\infty - 1/2 \rho (v_r)^2 (\text{Re}^2 + \Gamma^2) + \rho (v_\theta / r_0)^2 P z / r_0. \tag{5}
\]

According to (5), the pressure decreases as \( r \to 0 \). Since, as shown below, \( \Gamma^2 \gg \text{Re}^2 \) typically in practical applications, the pressure drop near the axis is mainly an effect of swirl. In physical terms, this pressure drop reflects the mutual counteraction of the centrifugal force and the radial pressure gradient—the so-called “cyclostrophic balance.” The pressure can also decrease or increase in the axial direction, depending on the sign of \( P \).

Thus, (2)–(5) represent a new five-parameter solution family. This number of parameters being quite large, the solution can capture a large variety of nontrivial flow patterns. A relevant adjustment of the parameters enables one to approximate (with a relative ease) some important elements of practical flows. As mentioned in the Introduction, prior solutions exist for flows over a porous cylinder\(^5^–^8\) and in an annulus;\(^9\) these are particular cases of our generalized solution (2) with \( \Gamma = 0 \). One can apply (2) without this restriction for a spiral flow in an annulus with porous walls that rotate and move along the axis. However, this use of (2) is overly restrictive and of limited interest. In this paper, we consider significantly more broad and interesting applications of both (2) itself, and also of composite solutions resulting from matching (2) and inner solutions. For the sake of these applications, we interpret (2) as a generalized vortex sink on surfaces of revolution.

### III. SHAPE OF THE SURFACES OF REVOLUTION

The family of surfaces of revolution is governed by (3a). Depending on the parameter values, the stream surface has a variety of shapes. Figure 1 shows typical meridional streamlines, i.e., the meridional sections of the stream surfaces. Because of symmetry with respect to the \( z \) axis, Fig. 1 shows only the right-hand sides. The shapes differ by number of extrema (which increases from left to right in Fig. 1), flow direction, and the asymptotic behavior as \( r \to 0 \). The latter feature is different for a strong sink [\( \text{Re} < -2, \) Figs. 1(a)–(c)], a weak sink \((-2 < \text{Re} < 0, \) Figs. 1(d)–(f)], and a source \([\text{Re} > 0, \) Figs. 1(g)–(i)]). Table I shows the representative parameter values for Fig. 1(a)–(i). To summarize, the stream surface shape depends on the radial flow rate \( \text{Re} \) and parameters \( a, b, \) and \( c, \) which characterize contributions to the axial flow from the uniform stream \( (a, \) radial advection \( (b, \) and axial pressure gradient \( (c, \) The radial advection causes accumulation and dispersion of the axial momentum for \( \text{Re} < 0 \) and \( \text{Re} > 0 \), respectively. The accumulation by sink flows \( \text{Re} < 0 \) leads to a singularity of the axial flow at \( r = 0 \); the character of this singularity is qualitatively different for strong \( \text{Re} < -2 \) and weak \( -2 < \text{Re} < 0 \) sinks.

### A. Strong sinks

For sink flows with \( \text{Re} < -2, \) \( z \to \pm \infty \) as \( r \to 0 \) along the surface of revolution, see (3a). Figures 1(a)–(c) correspond to different specific values of \( a, b, \) and \( c, \) but the same \( \text{Re} = -4 \) (this value is of a special interest, as shown in Sec. IV A). However, the features discussed below are common for any \( a, b, \) and \( c, \) for \( \text{Re} < -2 \). The shape of the surface of revolution depends on signs of \( a, b, \) and \( c. \) If \( a > 0, c > 0, \) and \( b < 0 \) in (3a), the axial flow is unidirectional. Streamlines start from \( z = \infty \) at \( r = \infty \) and go to \( z = - \infty \) as \( r \to 0 \) [Fig. 1(a)]; that is, a fluid particle descends on the surface of revolution along a spiral path. The velocities \( v_r \) and \( v_\theta \) monotonically increase (in modulus) downstream, in contrast to \( v_z, \) which has a (negative) maximum near the inflection point of the curve in Fig. 1(a), and tends to \( - \infty \) as \( r \to 0 \) and \( r \to \infty \).

If \( b > 0 \) and \( c > 0 \), then the surface has a shape like that shown in Fig. 1(b). There is a single extremum (minimum) at any value of \( a. \) A uniform flow influences only the radial position of the extremum. Since \( v_r \) is negative on the periphery but now positive near the axis, there is a counterflow, in contrast to the unidirectional flow in Fig. 1(a). The flow pat-
tern is even more complex for $c>0$, $b<0$, $a<0$, and sufficiently large $|a|$, which describes a "vortex sink with a barrier" [Fig. 1(c)]. As $a$ increases at fixed $b$ and $c$ (i.e., uniform downflow increases), the upflow disappears and the stream surface becomes that shown in Fig. 1(a).

Thus, the stream surface shape depends on the signs of $a$, $b$, and $c$, i.e. on the direction of each axial flow contribution: uniform flow ($a$), radial advection ($b$), and axial pressure gradient ($c$). The effect of ($c$) dominates the others at a large distance from the axis; see (2c) and (3a) as $r \to \infty$. Therefore, the axial flow direction for large $r$ is governed by the sign of $c$. Near the axis, ($b$) dominates; see (2c) as $r \to 0$ for $Re<0$. Therefore, the axial flow direction for small $r$ is governed by the sign of $b$. The uniform flow can dominate
for intermediate $r$ [see the vicinity of $r=1$ in Fig. 1(c)]. We interpret (in Secs. IV–V) the radial advection as a result of entrainment by a near-axis jet. This jet serves as a line sink for ambient fluid, with specific values of the radial flow rate (Re) that differ for weak and strong swirls (Sec. IV). This entrainment corresponds to a singularity in the contribution of $(b)$ to the axial velocity at $r=0$.

While the stream surfaces have a power-law singularity at $r=0$ for $Re<-2$, the singularity becomes logarithmic at $Re=-2$, and (3a) transforms into $\frac{\sqrt{r_0}}{\sqrt{r_0}+a(r/r_0)^2}+ b \ln(r/r_0)+c(r/r_0)^4$. As $r$ further increases, the stream surface becomes even more smooth.

### B. Weak sinks

For sink flows with $-2<Re<0$, $z$ has a bounded value at $r=0$; see (3a). This feature distinguishes Figs. 1(d)–1(f) from Figs. 1(a)–1(c), although the figures are otherwise similar. Figures 1(d)–1(f) correspond to different specific values of $a$, $b$, and $c$, but the same $Re=-1$ (this value is of a special interest, as shown in Sec. IV C). While $z$ is bounded at $r=0$, $\nu_z\to \pm \infty$ as $r\to 0$ along the surface in Figs. 1(d)–1(f), according to (2c). At $Re=0$, the singularity of $\nu_z$ becomes logarithmic, and (2c) transforms into $\nu_z=[W_z+W \ln(r/r_0)+W_p(r/r_0)^2]v/r_0$.

### C. Source

For source flows (Re$>0$), $\nu_z$ is bounded at $r=0$. Accordingly, the stream surfaces are tangential to $z=\text{const}$ planes at $r=0$, as Figs. 1(g)–1(i) show. Fluid particles move away from the symmetry axis, in contrast to Figs. 1(a)–1(f). The present classification captures all possible shapes of the stream surfaces. Simultaneous changes in the signs of $a$, $b$, $c$ result only in the reflection of streamlines with respect to planes normal to the axis, thus doubling the number of stream surface patterns.

Since the velocity is singular at $r=0$, solution (2) is not useful near the axis for practical flows. However, the vortex sink can serve as an outer solution, to be matched with an inner solution valid near the axis (i.e., near $r=0$). Below we discuss jet-like flows that are appropriate candidates for the inner solutions.

### IV. INNER SOLUTIONS

An inner solution is needed, in particular, to make $\nu_z$ bounded at $r=0$. On the other hand, the inner solution must rapidly decay as $r$ increases to match the outer solution. Therefore, $\nu_z$ has its maximal value near the axis, with jet-like flow in the inner ("core") region. The core radius is typically small in comparison with that of the outer region, where (2) is valid. For this reason, one can apply the boundary-layer approximation for the near-axis jet. The inner solutions can be swirl-free, weakly or strongly swirling jets, each of which is considered below.

#### A. Swirl-free jet

For a round swirl-free jet, the meridional flow inside the core can be represented by the Schlichting solution:

\[
\nu_z = 8 \nu_z^{-1} B [1+B(r/lz)^2]^{-2},
\]

\[
\nu_z = 4 \nu_z^{-2} B [1-B(r/lz)^2][1+B(r/lz)^2]^{-2},
\]

\[
\psi = -B r^2 (\ln(r/\rho_0)+c(r/\rho_0)^3)
\]

where the dimensionless constant $B = 3J/(64 \pi \rho_0^2)$ characterizes the axial momentum flux $J$ through a plane normal to the jet axis.

At fixed $z$, (6b) indicates that $r \nu_z/v \to -4$ as $B(r/lz)^2 \to \infty$. This exactly coincides with (2a) at $Re=-4$. At fixed $z$, (6a) indicates that

\[
\nu_z = 8 \nu B^{-1} \frac{r}{z^3} -4, \quad \text{as } B(r/lz)^2 \to \infty.
\]

(A1)

Compare with (2c), which gives

\[
\nu_z = W_p v \rho_0^3 r^{-4},
\]

at $W_p = W_0$ and $Re=-4$. Relations (A1) and (A2) have the same power-law dependence on $r$, but asymptote (A1) also involves $z$ (this difference is absent for strongly swirling jets, as shown in Sec. IV C). Although (A1) and (A2) are not matched uniformly with respect to $z$, their contributions rapidly decay as $r\to \infty$ and becomes negligible compared with the two first terms in (2c). Thus, (2) and (6) are matched with respect to $\nu_z$ but not with respect to $\nu_z$.

Note that it is typical that inner and outer solutions are not matched with respect to all velocity components in the leading terms. For example, the wall boundary layer and outer solutions are not matched with respect to the normal velocity (while the longitudinal velocity is matched). An example more closely related to our study is the Landau jet discussed in Sec. V A.

Concerning the streamfunction, (6c) yields the asymptotic expansion $\psi = -z(r_0+z^2(B/\rho_0)^{-1}r^{-2}+\cdots$ as $r\to \infty$ for fixed $z$. Note that the first term of the expansion coincides with the last term in (4). The second terms of the expansion and (4) have the same power law with respect to $r$ for $Re=-4$. The coefficient in the expansion depends on $z$ [similar to (A2)], but the difference with (4) is insignificant because both the second terms tend to zero and become negligible in comparison with $-z(r_0) r^{-2}$ as $r\to \infty$. Moreover, the other terms in (4) tend to infinity with increasing $r$.

Thus, the Schlichting jet is an appropriate inner solution, which can be matched with (2)–(4) at $Re=-4$. The Schlichting solution smoothes the singularity in (2) on the axis, $z>0$, with $\nu_z$ bounded at $r=0$, $\nu_z$ proportional to $r$, and $\psi \sim r^2$ in the vicinity of $r=0$. Note that the Schlichting jet is a
common part (near the axis) of many conical flows when the Reynolds number based on the axial velocity is large. Here we have shown the same feature for nonconical flows as well.

B. Weakly swirling jet

Concerning swirling jets as inner solutions, one can alternatively consider weak or strong swirl. By weak swirl, we mean that the meridional motion dominates the swirl, and the boundary layer equations for the meridional motion are decoupled from that for the swirl. For this reason, the meridional motion is governed by the Schlichting jet, and the boundary layer solution for the swirl is

\[ \nu_r = \Gamma v r^{r-1} \left[ 1 + z^2 / (B r^2) \right]^{1-1}. \]  

(7)

The asymptotic expansion of (7) as \( r \to \infty \) at a fixed \( z \) is \( \nu_r = \Gamma v r^{-1} - \Gamma v B^2 r^{-3} + \cdots \). The first term of the expansion coincides with the outer solution (2b). Note that the second term of the expansion has the same power law for \( r \) as the first term of (1b) at \( \text{Re} = -4 \). Therefore, the first term of (1b) is implicitly included in the inner solution (7). For this reason, we have omitted this term in the outer solution (2b).

The inner solution (6) involves a single new parameter \( B \) characterizing the jet intensity, while the circulation \( \Gamma \) in (7) is the same as that in the outer solution (2). Since we use the inner solution to replace the second term in (3), \( B \) plays the role of \( b \) in the combined solution studied in Sec. V A.

C. Long’s jet

The equations for the meridional motion and swirl are coupled when the swirl is strong, for which an appropriate inner solution is Long’s jet. Unfortunately, there is no general analytical solution for Long’s jet. However, the asymptotic behavior of Long’s solution is known.

\[ \nu_r \to -vr^{-1}, \]  

(8a)

\[ \nu_z \to -vr^{-1} \Gamma / \sqrt{2}, \]  

(8b)

and

\[ \nu_\phi \to -vr^{-1} \Gamma, \]  

(8c)

as \( r \to \infty \) at a fixed \( z \). The asymptote for the swirl (8c) agrees with (2b). Note that the next term in the expansion yields the difference, \( \nu_\phi - vr^{-1} \Gamma \), which vanishes exponentially as \( r \to \infty \); thus, the agreement is excellent.

Relations (8) show that the potential vortex is an adequate outer solution for Long’s jet. The asymptote for the radial velocity (8a) agrees with (2a) at \( \text{Re} = -1 \), which exactly corresponds to the weak-sink case [Figs. 1(d)–1(f)]. Concerning the axial velocity, (8b) coincides with the second term of (2c) at \( \text{Re} = -1 \) and \( W_\Gamma = \Gamma / \sqrt{2} \). We conclude that Long’s jet is an appropriate inner solution and can be matched with (2)–(4) at \( \text{Re} = -1 \). Note that the matching for \( \nu_z \) is uniform with respect to \( z \) here, in contrast to the weak swirl case. Thus (2) and Long’s jet are matched with respect to all velocity components.

D. Annular jet

A sufficiently strong swirl induces inversion of the near-axis flow. For weak swirl, a maximum of the longitudinal velocity (at fixed \( z \)) occurs on the symmetry axis. As the swirl increases, the maximum shifts away from the axis. The axis is now the position of a local minimum of the longitudinal velocity. This minimum decreases as the swirl increases and then becomes negative (\( \nu_z < 0 \)), i.e. flow reversal occurs. With a further increase in swirl, the \( \nu_z < 0 \) domain expands and the reversed flow becomes irrotational. The near-axis region of the irrotational flow is bounded by an annular swirling jet [see Fig. 3(a) for \( z > 0 \)]. Although the flow pattern developed is rather complex, there are analytical solutions for large swirl, i.e. \( \Gamma \gg 1 \). One of the solutions
covers only a thin near-axis region, while the other encompasses an outer (inviscid vortical) flow as well. The latter solution describes three flow regions: (i) \( \psi_1 = -\psi_1(1-x)/(1-x_j) \) and \( \psi_2 = 0 \), for \( x_j < x_i < l \) (near-axis flow); (ii) \( \psi_1 = -\psi_1 \tanh \xi \) and \( \psi_2 = \Gamma v r^{-1}(1-\tanh \xi)/2 \) in the vicinity of \( x = x_j \) (annular jet); (iii) \( \psi_1 = \psi_1 \tanh [x(2-(1+x_j)x_j)/x_j^2] \) and \( \psi_2 = \Gamma v r^{-1} \) for \( 0 < x < x_j \) (outer flow). Here \( \psi_1 = -\Gamma r_0^2 R x_j^2[1-x/(1-x)]/2 \). The anular jet direction is given by \( x_j = \cos \theta_a \); \( \theta_j \) is a polar angle of the annular jet. The above solution can be combined into the following compact form:

\[ \psi = \psi_1(1+\tanh \xi)/2 + \psi_2(1-\tanh \xi)/2, \]  

(9a)

\[ \nu_\phi = \Gamma v r^{-1}(1-\tanh \xi)/2; \]  

(9b)

this is a uniform approximation for the entire range, \( 0 < x_i < \xi \). A new control parameter introduced in (9) is \( x_j \); the other parameters are the same as those for the outer solution (2). The \( 1-x_j \) value must be small in order that (9) can serve as an inner (near-axis) solution for matching with (2). Suppose that \( 1-x_j \to 0 \) as \( \Gamma \to \infty \). Then the asymptotic relations (8) are valid for solution (9) as well and, therefore, (9) is well matched with (2).

To summarize, a few solutions describing near-axis jets can serve as an inner solution for the generalized vortex sink to obtain a composite solution that is regular on the axis of symmetry.

V. COMPOSITE VORTEX SINK

A. Matching approach

Here we match the exact NSE solution (2) obtained in Sec. II and the exact boundary layer solutions listed in Sec. IV to construct composite solutions that are uniform approximations in both the inner and outer flow regions. The composite solutions can be either multiplicative or additive. Consider a one-dimensional problem with a boundary layer at \( x = x_b \) as \( \text{Re} \to \infty \). Denote the outer solution as \( y_0(x) \) and the boundary layer (inner) solution as \( y_i(\eta) \), where the inner coordinate \( \eta \) is a scaled \( x \). The matching condition is \( y_i(\infty) = y_0(x_b) \). The multiplicative composite solution is \( y_m = y_0(x)y_i(\eta)/y_i(\infty) \) and the additive one is \( y_a = y_0(x) + y_i(\eta) - y_i(\infty) \).
It is instructive first to demonstrate how the matching technique works, using a simple exact solution. Consider an example closely related to our study—the Landau jet, where the Stokes streamfunction $\Psi$ has the representation

$$\Psi = \nu (r^2 + z^2)^{1/2} y(x),$$

$$y(x) = y_e = 2 \text{Re}_a (1 - x^2) / [4 + \text{Re}_a (1 - x)],$$

$$x = z (r^2 + z^2)^{1/2}$$

Here $\text{Re}_a = z v_a / \nu$, $v_a$ is the velocity at $r = 0$, and the subscript “$e$” indicates that $y_e$ is the exact solution (to distinguish from the following approximations).

As $\text{Re}_a \to \infty$, $y_e$ tends to the outer solution $y_0 (x) = 2 (1 + x)$ for $1 - x = O(1)$ and to the inner solution $y_i (\eta) = 4 \eta / (4 + \eta)$, $\eta = \text{Re}_a (1 - x)$ in the vicinity of $x = x_b = 1$. The inner solution is the Schlichting jet (6) (with $B = \text{Re}_a / 8$), and the outer solution describes a flow induced by the uniform sink along the axis, $x = 1$, with $y_e (\infty) = y_0 (1) = 4$ [=$\text{Re}$ in (2a)].

It is interesting that the multiplicative composite solution $y_m$ coincides with $y_e$ (for this particular example). The additive composite solution, $y_a = 4 \eta / (4 + \eta) - 2 (1 - x)$, does not coincide with $y_e$, but the error, $y_e - y_a = 8 (1 - x) / [4 + \text{Re}_a (1 - x)],$ uniformly tends to zero in $-1 \leq x \leq 1$ as $\text{Re}_a \to \infty$. Note that the outer and inner solutions, being matched with respect to $y_e$, are not matched with respect to $y_e$: $\nu_0 = 2 \nu / (r^2 + z^2)^{1/2}$ for the outer solution providing $\nu_0 = 2 \nu / |z|$ at $x = 1$, while $\nu_0 \to 0$ as $\eta \to \infty$. Thus, this example shows that matching with respect to velocity components is not generic but exceptional. The Long and annular jets are such exceptions because $\nu_z$, $\nu_r$, and $\nu_\phi$ in these cases have the same asymptotic behavior ($\sim r^{-1}$ as $r \to \infty$), as shown in Secs. IV C and D.

Now we apply this matching technique to obtain composite uniform approximations, using (2) as the outer solution and the near-axis jets (Sec. IV) as inner solutions. The following additive composite solutions result from summarizing the outer and inner solutions and extracting common parts,18 which are here singular terms in the generalized vortex sink.

**B. Weakly swirling combined flow**

Solutions of Sec. IV are unidirectional jets originating at $z = r = 0$ and flowing in the positive $z$ direction, thus covering only the half-axis. However, the generalized vortex sink is singular on the entire $z$ axis, thus necessitating an inner solution encompassing the entire axis. Therefore, to match the vortex sink, one has to construct a solution describing a jet flowing from its origin in both the positive and negative $z$ directions; this is the bipolar jet. For this goal, we apply (6) for both positive and negative $z$, substitute (6) for the second term in (4), and thus obtain

$$\psi = a (r r_0)^2 - B r^2 (z r_0)^{-1} [1 + B (r z)^2]^{-1} + c (r r_0)^6.$$  \hspace{1cm} (10)

The solution (10) remains singular at the origin, $z = r = 0$. To avoid this singularity, one needs to match (10) and a stagnation flow near $z = r = 0$. This regularization needs a special study that is out of the scope of this paper. Here we ignore the singularity at the origin because it is of minor influence for the global flow pattern. The swirl distribution (7) is an even function of $z$, so we apply (7) for the entire flow region.

It is striking that the simple composite solution (10) describes rather complex flow patterns as Fig. 2 illustrates, where streamlines of the meridional motion and the corresponding profile of $\nu_e (r)$ at $z = -\infty$, resulting from (10), are shown for the parameter values listed in Table II.

First, consider the case where there is no uniform axial component in the outer flow (i.e., $a = 0$), but there is a shear flow induced by the axial pressure gradient ($c \neq 0$). In this case, the bipolar jet extends to infinity in both $z$ directions [Fig. 2(a)], a flow pattern termed by us a “bipolar jet in a shear flow.”

Figure 2(b) shows the case for uniform axial flow ($c = 0$). Since the uniform flow is directed upward, it stagnates the downward jet at some negative $z$ [see the saddle point in the lower panel of Fig. 2(b)]. As a result, a recirculatory “bubble” develops between the saddle point and the origin in Fig. 2(b). In contrast, the upward jet extends to infinity because the uniform flow acts to increase the jet velocity. The flow pattern in Fig. 2(b) is thus a “bubble in a uniform flow.”

If there is also a shear in the outer flow, the bubble then changes its shape. When the uniform and shear parts of the outer flow have the same direction, the uppermost part of the bubble becomes narrow in comparison with the downstream part [Fig. 2(c)]. The bubble shape depends on values of $a$ and $b$ [both are negative in Fig. 2(c)], but the flow topology remains the same in Figs. 2(b) and 2(c).

The flow topology becomes significantly more complex when the uniform and shear parts of the outer flow have opposite directions. The flow pattern in this case changes qualitatively with the ratio $c/a$. In Figs. 2(d)–2(f), $a (c < 0)$ is fixed but $c (c > 0)$ increases. For sufficiently small $c$, the meridional flow near the axis is similar to that at $c = 0$ [compare Figs. 2(d) and 2(b)]. However, the axial flow has an opposite direction far from the axis in Fig. 2(d). Between the upward and downward flows, there are two semi-infinite domains of recirculatory flow. The existence and arrangement of recirculatory domains are important for effective heat and mass transfer in practical flows (e.g., in combustion chambers, see Sec. VII D). These domains are separated by a saddle circle (denoted by crosses) in Fig. 2(d). The flow in the upper domain moves down at the periphery, turns around near the saddle circle, and moves up for smaller $r$. The flow is similar (but opposite) in the lower domain. There are two stream surfaces passing through the saddle circle and separating the unidirectional and recirculatory flows.

As $c$ increases, the saddle circle approaches, touches, and destroys the bubble surface. The passing of the circle through the surfaces is a boundary crisis or catastrophe that changes the flow topology. The stream surfaces reconnect and the bubble [Fig. 2(d)] transforms into the vortex ring [Fig. 2(e)]. The semi-infinite recirculatory domains now extend up to the axis and the upward unidirectional flow disappears [compare Figs. 2(d) and 2(e)]. There is a gap between the recirculatory domains where the vortex ring is located in Fig. 2(e). To classify flow patterns, we call the...
flow shown in Fig. 2(e) a "vortex ring in a counterflow," much like an axisymmetric jet flow.

As \( c \) increases further, the saddle circle mentioned above approaches the other circle (the centerline of the vortex ring). At some \( c \), these two circles merge and cease to exist, and the vortex ring disappears. For larger \( c \), the flow pattern becomes that shown in Fig. 2(f). Now, the outer downflow penetrates the gap between the upper and lower recirculatory domains. There is a bipolar jet near the origin as in Fig. 2(a), but the downward jet exhibits a vortex breakdown-type structure with a semi-infinite recirculatory domain. The flow in Fig. 2(f) is thus a "bipolar jet with vortex breakdown."

FIG. 2. Meridional streamlines for the generalized vortex sink matched with the bipolar Schlichting jet. (a) No uniform outer flow. (b) Uniform outer flow. (c) Codirected outer uniform and shear flows. (d–f) Counterdirected outer uniform and shear flows; the shear flow part increases from (d) to (f). See Table II for parameter values.
As $a \to 0$ for fixed $c$ (or as $B$ increases), the lower recirculatory domain in Fig. 2(f) descends, so the flow pattern approaches that shown in Fig. 2(a).

In summary, Fig. 2 shows a variety of flow patterns described by (10), for which the swirl distribution is similar and rather simple. According to (7), the circulation is constant along the conical surfaces, $r/z = \text{const}$. As the polar angle $\theta$ increases from $0^\circ$ to $90^\circ$, the circulation grows from zero on the axis, $r=0$, to its maximum value on the plane, $z=0$.

C. Strongly swirling combined flow

Despite the fact that strong swirl influences the meridional motion, the global flow structure in this case is nearly the same as that for weak swirl, unless the bipolar jet becomes annular. The small difference is in the $v_r$ distribution near the axis; in particular, the maximum $v_r$ position can shift away from the axis (see Secs. IV C and IV D) for Long's jet, which we choose not to address here. Consideration of Long's jet is technically laborious due to the absence of an analytical solution, and also would not provide qualitatively new flow patterns in comparison with those shown in Fig. 2. For this reason, we consider here the composite solution combining (2) and the analytical solution (9) for the annular jet, providing new qualitative features in the flow structure.

First, we construct the bipolar jet starting from solution (9) that is valid for $z>0$ only. Denoting the streamfunction (9a) as $\psi_+$, we define $\psi_-= -\psi_+(z)$, and introduce the combined inner solution,

$$\psi_i = \psi_+, \quad \text{for } z>0 \quad \text{and} \quad \psi_i = \psi_-, \quad \text{for } z<0,$$

which is an odd function of $z$. Substituting $\psi_i$ for the second term in (4) yields the composite solution for the streamfunction:

$$\psi = a(r/l_{a})^2 + \psi_i + c(r/l_{b})^4. \quad (11)$$

Since the swirl is symmetric with respect to $z=0$, the composite solution for the swirl is

$$v_\theta = v_\theta^+, \quad \text{for } z>0 \quad \text{and} \quad v_\theta = v_\theta^-, \quad \text{for } z<0,$$

where $v_\theta^+$ is defined by (9b) and $v_\theta^-(z) = v_\theta^+(1/z)$ for $z < 0$.

Figure 3 shows typical flow patterns (for particular values in Table III) for the strongly swirling flows with annular jets [the profile $v_z(r)$ at $z=\infty$ is shown in the top panels]. The matching procedure is valid if the polar angle $\theta_1$ of the annular jet is small. We use sufficiently large $\theta_1$ for Fig. 3 to highlight the flow patterns; one can imagine that the near-axis region with the annular jet is "blown up."

In Fig. 3(a), the annular bipolar jet itself is shown without any outer flow, i.e. $a = c = 0$ in (11). Fluids flow to the origin along both the axis and equatorial plane. This flow collision induces a bipolar jet that flows outward along a conical surface, $\theta = \theta_1$. Such a flow can be induced by boundary conditions corresponding to a given vortex sink on the plane $z=0$. The fact that the jet is annular, not consolidated, is owing to the strong swirl.

Figure 3(b) shows the resulting flow due to the above jet and a uniform downflow ($c=0, a>0$). The uniform stream simply increases the downflow near the upper half-axis [Fig. 3(a)], because these flows are codirected. In contrast, the upward annular jet is directed oppositely to the uniform stream. Since the maximal jet velocity is proportional to $1/r$ with a uniform stream velocity, the uniform stream dominates the jet for sufficiently large $r$. For this reason, the stream turns down and inward, forming a vortex-ring flow near the origin. The vortex-ring domain is separated from the ambient flow by a surface whose section resembles a homoclinic orbit in Fig. 3(b). This orbit, starting at the origin, is first directed upward along the conical surface, then turns downward and comes back to the origin tangentially along the $z=0$ plane.

The uniform stream and the jet-induced upflow [Fig. 3(a)] are oppositely directed near the lower half-axis in Fig. 3(b). Since $v_z$ of the upflow decays proportionally to $1/z$ as $z \to -\infty$, the uniform stream dominates the upflow sufficiently far below, where the fluid flows downward. However, the upflow dominates the uniform flow near the origin, and the fluid flows upward. For this reason, there is a saddle point on the negative $z$ axis [Fig. 3(b)] and a "bubble" appears.

Figure 3(c) shows the opposite case to that in Fig. 3(b). That is, there is an outer downflow with shear, but without the uniform component (i.e., $a = 0, c>0$). The main difference with the pattern in Fig. 3(b) is the bubble shape. The bubble in the lower part of Fig. 3(c) extends to infinity because the outer downflow now has zero velocity on the symmetry axis. For this reason, the downflow cannot stagnate the jet-induced upflow [see the near-axis region for $z<0$ in Fig. 3(a)] for finite $z$. Since the upflow velocity decays as $1/\sqrt{z}$, the width of the upflow tends to zero as $z \to -\infty$. Also, the width of the recirculatory domain [positioned below the origin in Fig. 3(c)] tends to zero as $z \to -\infty$. In this case, there is a recirculatory "tail" [Fig. 3(c)] instead of a "bubble" [Fig. 3(b)] for $z<0$.

In Fig. 3(d), we show a flow pattern that is intermediate between those in Figs. 3(b) and 3(c), containing both the uniform and shear downflows ($a>0$ and $c>0$). The difference is that the lower part of the bubble is flattened in Fig. 3(b), but rounded in Fig. 3(d). The flat bubble shape in Fig 3(b) is due to the fact that the upflow induced by the annular jet [Fig. 3(a)] is nearly uniform with respect to $r$ at a fixed $z$. This feature of the irrotational flow near the axis is evident from (9). Since the ambient flow is also uniform, the bound-

<table>
<thead>
<tr>
<th>Figure</th>
<th>$a$</th>
<th>$B$</th>
<th>$c$</th>
<th>$\psi$ (contour levels)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>100</td>
<td>−0.042</td>
<td>−5, −2, 0, 2, 5, 10, 20</td>
</tr>
<tr>
<td>(b)</td>
<td>−0.4</td>
<td>4</td>
<td>0</td>
<td>−5, −2, −0.5, 0, 0.1, 0.25, 0.5, 0.6</td>
</tr>
<tr>
<td>(c)</td>
<td>−0.4</td>
<td>4</td>
<td>−0.1</td>
<td>−5, −2, −0.5, −0.1, 0, 0.05, 0.25, 0.5</td>
</tr>
<tr>
<td>(d)</td>
<td>−0.4</td>
<td>4</td>
<td>0.005</td>
<td>−10, −5, −2, −2, −0.5, 0, 0.1, 0.25, 0.5, 0.6, 2, 5</td>
</tr>
<tr>
<td>(e)</td>
<td>−0.4</td>
<td>4</td>
<td>0.012</td>
<td>−10, −5, −2, −2, −0.5, 0.1, 0.36, 0.5, 0.6, 2, 5, 10</td>
</tr>
<tr>
<td>(f)</td>
<td>−0.4</td>
<td>4</td>
<td>0.04</td>
<td>−10, −5, −2, −2, −0.2, −0.05, 0, 0.1, 0.36, 0.5, 0.6, 2, 5, 10</td>
</tr>
</tbody>
</table>
ary between the upflow and downflow in the lower part of Fig. 3(b) is nearly horizontal near the axis. This boundary becomes parabolic in Fig. 3(d), owing to the shear outer flow. The pattern in Fig. 3(d) approaches that in Fig. 3(c) if \( a \to 0 \) with fixed \( c \), and that in Fig. 3(b) if \( c \to 0 \) for fixed \( a \).

The flow patterns are significantly more complicated if the uniform and shear components of the outer flows have opposite directions (\( ac < 0 \)). Figures 3(e) and 3(f) show the...
patterns for the same shear upflow \((c<0)\) but different uniform downflows \((a>0)\). For the slower downflow [Fig. 3(e)], the pattern for \(z>0\) is rather simple. There is a near-axis recirculatory domain separated from the ambient upflow by the heavy lines in Fig. 3(e). Inside the domain, there is the vortex ring near the origin. On the upper part of the vortex-ring boundary, there is the saddle circle [denoted by the \(\times\) symbols near the top of Fig. 3(e)]. Beyond the circle, there is a semi-infinite recirculatory region (not shown).

Below the origin \((z<0)\), there is a cone-shape bubble surrounded by the wide vortex ring. Between the bubble and ring, there is a thin gap that separates also the ring from the outer upflow. The flow inside the gap starts at \(z=-\infty\), moves up, then around the ring, and finally down near the axis. Between these regions of up- and downflows, there is an additional recirculatory semi-infinite region, which touches the ring along the saddle circle. A complex arrangement and multiplicity of recirculatory domains, as shown in Fig. 3(e), is typical in many technological swirling flows, e.g. in the Ranque–Hilsch tubes (Sec. VII C) and vortex combustion chambers (Sec. VII D).

The qualitative difference between Fig. 3(f) and Fig. 3(e) is in the upper part of the flow \((z>0)\). The saddle circle (the \(\times\) symbols) is located on the boundary separating the outer upflow and the recirculatory domains in Fig. 3(f). In contrast to Fig. 3(e), the outer flow does not extend up to the origin in Fig. 3(f). The “gap” recirculatory domain is attached to the upper vortex ring along the “heteroclinic trajectory,” which starts at the upper saddle and terminates at the origin [Fig. 3(f)]. The other domains are similar in Figs. 3(e) and 3(f); the lower vortex ring is simply larger in Fig. 3(f).

In summary, the results of this section show that the composite solutions resulting from matching of the generalized vortex sink with jet-like inner solutions describe a rich variety of rather complex flow patterns, typical of practical swirling flows and important for momentum, heat, and mass transfer. For example, there are seven different flow domains in Figs. 3(e) and 3(f). It is striking that such complicated flows are encompassed by our rather simple analytical solutions, which explicitly reveal the physical effects responsible for a variety of flow separation phenomena. Namely, radial advection, swirl, and counterflow in various combinations induce “bubbles,” vortex rings, and semi-infinite recirculatory domains. Such flow elements are important features of swirling flows in nature and technology, as demonstrated in Secs. VI–VII for particular applications.

### VI. APPLICATIONS OF THE GENERALIZED VORTEX SINK

#### A. Whirlpool

A well-known solution exists for a whirlpool in an inviscid fluid;\(^{19}\) our new results include a solution for a viscous fluid and an analogy drawn between outer flows of the Schlichting jet and whirlpools.

To model a whirlpool, e.g., on the ocean surface, one must account for gravity, which causes the last term in (5) to be \(-pgz\). Here, \(g\) is the acceleration due to gravity, and \(z<0\) below the ocean level. On the whirlpool surface \(z=f(r)\), the pressure is constant and equal to the atmospheric pressure \(p_a\). Substituting (3a) with \(z_0=0\) in (5), and satisfying the requirement that the sum of the two last terms in (5) is zero, we find \(a=c=0\), \(Re=-4\), and \(b=-\frac{1}{2}(16+\Gamma^2)\). This requirement physically means that the dynamic head, \(-\frac{2}{\rho}v(\nabla v)^2(\nabla^2+\Gamma^2)\), balances the hydrostatic pressure \(-gz\) on the whirlpool surface \(z/r_0=b(r/r_0)^{2-\Gamma}\).

If the swirl Reynolds number is sufficiently high, i.e. \(\Gamma^2>16\), then \(b=-K^2/(8\pi^2\rho r_0^2)\) becomes independent of viscosity. For example, for \(\nu_0=0.001\) m/s at \(r=0.2\) m and \(\nu=10^{-6}\) m\(^2\)/s (values relevant for a bathtub vortex), \(\Gamma=200\) and the viscous contribution to \(b\) is negligible, even in this case. Since the centrifugal acceleration at \(r=r_0\) is \(g_{c0}=K^2/(4\pi^2r_0^2)\), the latter expression for \(b\) can be written as \(b=-\frac{2}{\rho_0}g_{c0}/g\). Therefore, the whirlpool shape is determined by the centrifugal/gravitational acceleration ratio at \(r=r_0\). This coincides with the corresponding solution for an inviscid fluid.\(^{19}\)

For a whirlpool to develop, the centrifugal acceleration must be of the same order of magnitude as \(g\). In Fig. 4, we choose \(g_{c0}=\frac{g}{2}\), i.e. \(b=-0.25\), and \(\Gamma=200\) \((S=-50)\) to illustrate a physically realizable whirlpool [employing (2)–(4)]. The corresponding stream surfaces \(\psi=0\), 1, and 2 (left side) and the axial velocity profile (right side) are shown in Fig. 4(a).

In Fig. 4(b), a streamline on the whirlpool surface \(\psi=0\) explicitly reveals the spiral character of the flow. The pitch increases downstream because the axial velocity dominates the swirl as the streamline approaches the axis, although both velocity components tend to infinity.

The velocity field follows from (2) with \(Re=-4\), \(\Gamma=-4S\), \(W_s=8b\), and \(W_c=0\). The whirlpool attracts ambient fluid to the axis, and the radial flow rate, \(Re=-4\), is exactly the same as that for the Schlichting round jet (see

---

### TABLE III. Parameters for Fig. 3.

<table>
<thead>
<tr>
<th>Figure</th>
<th>(a)</th>
<th>(\Gamma)</th>
<th>(c)</th>
<th>(x_f)</th>
<th>(\psi\times10^{-2}) (contour levels)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>50</td>
<td>0</td>
<td>0.9</td>
<td>2, 1.5, 1, 0.5, 0.25, 0.25, 0.5, 1, 1.5, 2</td>
</tr>
<tr>
<td>(b)</td>
<td>5</td>
<td>50</td>
<td>0</td>
<td>0.9</td>
<td>0.4, 0.3, 0.1, 0.03, 0.01, 0.25, 0.5, 1, 1.5, 2</td>
</tr>
<tr>
<td>(c)</td>
<td>0</td>
<td>50</td>
<td>0.5</td>
<td>0.9</td>
<td>-0.5, -0.2, -0.1, -0.05, 0, 0.25, 1, 3</td>
</tr>
<tr>
<td>(d)</td>
<td>8</td>
<td>50</td>
<td>1</td>
<td>0.9</td>
<td>-0.2, -0.05, -0.02, 0, 0.02, 0.25, 0.5, 1, 2</td>
</tr>
<tr>
<td>(e)</td>
<td>10</td>
<td>50</td>
<td>-0.5</td>
<td>0.9</td>
<td>-10, -5, -1, -0.015, 0, 0.05, 0.1, 0.5, 0.9, 1.4</td>
</tr>
<tr>
<td>(f)</td>
<td>15</td>
<td>50</td>
<td>-0.5</td>
<td>0.9</td>
<td>-5, -1, -0.1, -0.01, 0, 0.05, 0.1, 0.5, 1, 1.5, 2.1</td>
</tr>
</tbody>
</table>
Sec. IV A). This coincidence of entrainment for the Schlichting jet and whirlpool is a new result that has a clear physical basis. The entrainment of the fluid remote from the axis is the same viscous effect, despite the different sources of axial momentum in the Schlichting jet and whirlpool. Note that Schlichting’s solution describes the boundary-layer limit for a strong jet and also that the axial velocity on the whirlpool surface tends to infinity as \( r \to 0 \). Therefore, in both cases, there is a strong axial flow attracting ambient fluid. Another reason for the coincidence of Re for the whirlpool and the Schlichting swirl-free jet is the fact that the axial velocity dominates the swirl as \( r \to 0 \) along the whirlpool surface (because \( v_g \sim r^{-1} \) and \( v_z \sim r^{-4} \)), so that the swirl does not influence the entrainment rate.

Thus, our model captures the whirlpool shape, for which the role of viscosity is typically negligible, but also the entrainment an ambient fluid—-a viscous effect.

B. Cosmic jets

The formation of massive cosmic objects such as stars, galaxy cores, and “black holes” generates far-range bipolar jets. The jets develop near accretion disks, which are regions of higher density than that of the ambient media. The disk matter moves as a planar vortex sink condensing on a massive object. This motion drives the ambient medium and induces normal-to-disk bipolar swirling jets, which advect the angular momentum far away from the disk. The axial momentum is focused by the converging flow, but dispersed by viscous diffusion.

A simple hydrodynamic model describes the farfield of these jets, where the dispersion dominates and the axial velocity decays with distance. In the near field, the focusing dominates and the axial velocity increases along the axis from zero on the disk plane, \( z = 0 \). In an intermediate domain, where the axial velocity reaches its maximum, the focusing and dispersion nearly balance each other, and the axial velocity is nearly independent of \( z \). In Fig. 5, we show that the solution (2) models the bipolar swirling jet in this domain, using the same parameter values as for Fig. 4.

The physical reason for specifying \( c = 0 \) is that there is no axial pressure gradient in the ambient space; we take \( a = 0 \), to satisfy the condition that \( v_z \) tends to zero as streamlines approach the disk. Finally, we choose \( Re = -4 \) as the maximum entrainment rate. Cosmic jets are certainly turbulent, but the induced flow of an ambient medium can be laminar far from the axis. Also, a uniform eddy viscosity is a satisfactory model for the mean flow of turbulent jets, so that \( Re \) still equals \(-4\), but with a \( Re \) based on the eddy viscosity. The parameters \( b \) and \( S \) remain free and describe the gravity of the massive body and the angular momentum of disk particles, respectively.

The specific values chosen for Fig. 5 show that the same parameter values can illustrate different physical applications. In particular, the streamlines of the meridional motion for the cosmic jet model are the same as shown in Fig. 4(a),
but now the flow pattern is symmetric with respect to plane \( z=0 \) (Fig. 5). The accretion disk corresponds to plane \( z=0 \), and the massive body is located at the frame origin in this model.

In contrast to the whirlpool case, the pressure is not constant on stream surfaces and the fluid occupies the entire space. The common features are that stream surfaces converge to the axis downstream and the axial velocity increases along streamlines (compare with the farfield of the jet, \(^1\) where stream surfaces diverge from the axis and the axial velocity decays downstream).

Thus, solution (2) models an ambient flow induced by a strong jet, regardless of how the jet is forced. The physical mechanism of forcing is quite different for the whirlpool and cosmic jets, but the kinematic features are the same: (i) flow focusing by the radial convergence and (ii) irrotational character of swirl in an outer flow. An additional illustration of such universality is a tornado model, considered below.

C. Tornado

A solution similar to that shown in Figs. 4 and 5 can also model a swirling flow near a wall outside a boundary layer. As an example, consider the flow produced in a cylindrical container by a rotating endwall.\(^{21}\) Our model approximates well the flow near the axis in the vicinity of the fixed endwall, where the flow pattern is similar to that in tornadoes. Note that our model is not valid near the rotating endwall, because it describes potential but not solid-body swirl. Another example is a tornado itself or, more precisely, the region adjacent to the ground where the tornado converges to its axis (conically similar flows model the farfield of a tornado\(^1\)).

It has been observed\(^{22}\) that a tornado funnel often abruptly expands into a wide bulge at some height from the ground. Bulge formation in tornadoes can be caused by different factors. Suppose, for example, that a tornado funnel enters a region of high atmospheric turbulence. Turbulent mixing decreases the near-axis swirl, thus increasing the pressure by cyclostrophic balance. In turn, the induced positive axial pressure gradient slows the axial flow and causes streamline divergence, i.e., positive values of the radial velocity.

In terms of our model, this pressure increase and divergence correspond to \( P>0 \) and \( \text{Re}>0 \), and solution (2) predicts an abrupt widening of the funnel (Fig. 6). Figure 6(a) shows the meridional streamlines, which model the expansion of the tornado funnel into a bulge; Fig. 6(b) illustrates tornado formation near the ground. The parameters are \( a=10, b=10, c=-10, \text{Re}=4, \text{and } S=-50 \) for Fig. 6(a) and \( a=c=0, b=-0.25, \text{Re}=-4, \text{and } S=50 \) for Fig. 6(b). These values are chosen to satisfy the condition that \( \nu_r \to 0 \) as \( r \to \infty \) along streamlines approaching the ground, to obtain a strong jet, and to match the flow characteristics in Figs. 6(a) and 6(b).

In summary, our solution (2)–(5) is able to approximate both tornado focusing and expansion into a bulge. The latter effect leads us to the general problem of vortex breakdown. First, we consider vortex breakdown without reverse flow, which we call a ‘‘vortex-breakdown wake.’’

D. Vortex-breakdown wake

The appearance of the tornado bulge is certainly a type of vortex breakdown. There exists a conical model\(^1\) for vortex breakdown at a tornado’s mid-height, with a reversed flow inside the bulge. In contrast, there is no flow reversal in the above model for the bulge. Vortex breakdown is indeed observed without reversed flow under some conditions.\(^{23}\) In this case, stream surfaces gradually diverge as in a far slender wake. At very high Reynolds numbers, a slender turbulent wake develops just downstream of the vortex breakdown point.\(^{24}\)

For slender wakes, the longitudinal velocity has the same sign on and far away from the axis, due to viscous diffusion of axial momentum. Thus, flow reversal, which occurs inside a vortex-breakdown ‘‘bubble,’’ does not occur in a slender wake.

There is a recent experimental observation\(^{25}\) of vortex breakdown in a cylindrical container with a thin rod on the
axis rotating independently of the rotating endwall. In Fig. 7(a), the rod is seen as a bright vertical strip. Near the rod, the flow moves downward from the fixed endwall to the rotating endwall. If the rod is fixed, then there is a recirculatory "bubble" similar to that appearing without the rod.21 However, if the rod rotates in the same direction as the endwall, the bubble transforms into a slender domain, starting behind the breakdown point and extending up to the endwall.

The rod "forces" the meridional flow to stagnate, but the swirl is fixed and independent of $z$ near the rod due to the no-slip condition. Near the lower part of the rod, there are contributions to the meridional flow from two effects: (i) the rotating disk generates a flow toward it near the axis, and (ii) the interaction of the rotating rod normal to the fixed wall, inducing a flow away from the wall (see Ref. 11, p. 93 and p. 214, respectively). Since these effects oppose each other, we expect a very slow meridional motion inside the slender domain.

Figure 7(a) shows a photo of the breakdown domain;25 note the light diverging strips. If the rod corotates fast enough (the rod/endwall angular velocity ratio is $\approx 6$), the breakdown "bubble" is transformed into a slender wake. We model the domain shape with the help of solution (3) at the parameter values chosen to approximate the experimental observation. Figures 7(b) and 7(c) show the meridional cross section of the stream surface and a three-dimensional view of a streamline on this surface that models the wake surface of Fig. 7(a). The parameters: $a = 12$, $b = 10$, $c = -10$, $Re=4$, and $S=100$, provide good qualitative agreement with the experimental breakdown shape (we cannot compare the velocity field because the corresponding experimental data are not yet available). The experiment shows that the wake surface expands downstream of the breakdown point, develops into a nearly cylindrical shape, and then again abruptly expands. Our model captures even this fine feature of the wake surface.

To summarize this section, even the (rather simple) generalized vortex source itself is useful in modeling many practical flows. In the next section, we apply our composite solutions to model flows that exhibit considerably more complex patterns.

VII. APPLICATIONS OF THE COMPOSITE SOLUTIONS

A. Vortex breakdown bubbles

Although there is no reverse flow in the above examples, the composite solutions (10) and (11) enable one to model reversing swirling flows, including vortex-breakdown "bubbles."

Figure 8(a) is a photo of the typical vortex breakdown bubble observed experimentally.21 In Figs. 8(b) and 8(c), we compare the flow resulting from (10) with $B=1$ and for both Figs. 8(b) and 8(c), but with $a = -0.08$ and $c = 0$ for (b), and $a = -0.08$ and $c = -0.0001$ for Fig. 8(c). These parameter values provide the best agreement with the experimental result.

B. Self-swirling flow in meniscus

An intriguing flow phenomenon that we model here by solution (10) is the self-swirling observed in liquid menisci of electrosprays.26,27 A drop of an electrically conducting liquid at the edge of a capillary tube takes a conical form in
the presence of an axial electric field. Figure 9(a) is a schematic showing the capillary tube $T$, meniscus $M$, and electric field $E$.

Electrical shear stresses acting on the meniscus surface [$\tau$ in Fig. 9(a)] induce a recirculatory motion of the liquid that flows to the apex near the surface and away from the apex along the axis [$C$ in Fig. 9(a)]. Under some conditions (a sufficiently strong electric field, and small viscosity and electrical conductivity of the liquid), swirl appears, although there is no obvious swirl forcing. This swirl can become very strong, so that the flow inside the meniscus resembles a tornado. Far away from the apex, the flow is satisfactorily approximated by conical solutions.26,27 Near the apex, however, the meniscus shape is not conical, and a jet erupts from the tip [$J$ in Fig. 9(a)]. These features are not covered by the conical solutions, but can be modeled with the help of solution (10). Figures 9(b)–9(d) are obtained with (10) for $a = 0$, $B = 1000$, $c = -0.001$, and $S = 50$. Figure 9(b) shows the meridional streamlines, and Figs. 9(c) and 9(d) show the three-dimensional view of the streamlines. Figure 9(c) shows a streamline embedded in the meniscus and positioned at some distance from the surface [$C$ in Fig. 9(a)]. The formation of a submerged swirling jet directed inside the meniscus is clear in Figs. 9(b) and 9(c). Figure 9(d) shows a streamline on the surface of both the meniscus and the jet [$M$ and $J$ in Fig. 9(a)]. The parameters are adjusted to obtain good agreement with the experimentally observed flow patterns27 (only visualization with the help of a microscope is currently available for the flow near the meniscus tip). Note that the near-axis parts of the streamlines in Figs. 4 and 9 differ. As mentioned above, the pitch increases downstream in Fig. 4(b), but is nearly constant in Figs. 9(c) and 9(d). The reason is that stream surfaces diverge downstream in the Schlichting jet (Fig. 9), but converge for solution (2) (Fig. 4). The axial velocity is bounded and decays as $1/z$, thus influencing the pitch defined by the $v_z/v_\phi$ ratio. These flow features in menisci are quite similar to those in vortex tubes, as discussed in the next section.

C. Vortex tube

The Ranque–Hilsch vortex tube is in widespread use as a compact, environmentally safe cooling device containing no moving parts.28 Figure 10(a) shows the schematic of stream surfaces in a meridional cross section (not to scale). Compressed air enters the tube tangentially through inlet $I$, generating a strong swirling flow. This flow then separates into two streams:14 (i) stream $C$ (to become cold at the outflow) reverses direction near the stagnation point $S$ and leaves the tube through orifice $OC$, and (ii) stream $H$ (that becomes hot at the outflow) is unidirectional and leaves the tube far downstream through the (typically peripheral) outlet $OH$.

Our solution family (2)–(3) enables one to model streams $C$, $H$, and the separating surface $S$ through an appropriate choice of the parameters. Figure 10(b) shows the model streamlines of the cold $C$, hot $H$, and separating $S$ flows for $a = 0.5$, $c = 0$, $Re = 4$, and $S = 50$. The streamlines differ by values of $b = 0.25 (C)$, $0 (S)$, and $0.25 (H)$. The $b$ sign governs whether the flow is reversing, separating, or unidirectional.

Solution (10) provides a better approximation of the flow and captures all the streams simultaneously in a single flow field. Figure 10(c) shows streamlines of the meridional motion calculated from (10) using $a = 0.5$, $B = 1000$, and $c = 0$. The numerical values are chosen here just to demonstrate the ability of our solution to model a flow with very different scales in different regions. The goal of the present work is not to provide specific numerical data, but to clearly illuminate physical processes which are crucial for the Ranque–Hilsch effect. Analysis of the flow field details and optimization of the Ranque–Hilsch effect through parameter changes are the subject of a separate study.

Our model captures two important features of the flow dynamics in the vortex tube: (i) focusing of the swirling flow to the axis along separatrix $S$, and (ii) a sharp decrease in pressure along streamlines according to (5). The pressure decrease provides adiabatic cooling of air ($\gamma$ is the ratio of specific heats), while the swirl focusing generates a sharp peak of the axial vorticity near the axis.

The effects (i)–(ii) are the basis for the next two: (iii) strong diffusion of the vorticity causing kinetic energy transfer from near-axis to peripheral regions, and (iv) separation of near-axis and peripheral flows. Our view (based on the analysis of the literature and our own preliminary studies) is that features (i)–(iv) are the essence of the Ranque–Hilsch effect. The present model describes all these features and provides a background for more detailed studies and optimization.
A solution for the heat equation (based on our results for velocity and pressure) can also be analytically calculated. This is beyond the scope of this paper, which addresses fluid motion only. Nevertheless, in the next section, we consider one "thermal" application that does not necessitate a solution of the heat equation.

D. Vortex burner

It is well known that swirling flows with vortex breakdown provide flame stabilization and are useful for combustion. Figure 11(a) shows a schematic of an ABB EV-burner. Tangential injection of gaseous fuel and air through slits in the conical sidewall generates a strongly swirling flow inside the burner. Liquid fuel is injected through the central nozzle, then atomized into drops and evaporated. Vortex breakdown induces the recirculatory zone $RZ$ (near the exit), which stabilizes the flame front located slightly upstream and is shown as a thin parabolic shell. For more details concerning the burner design and flow features see Ref. 29. Here we present a simplified flow model based on our solution to demonstrate how the flame front can be calculated near the vortex breakdown "bubble."

Figure 11(b) shows a flow pattern obtained with (10) for $a = -0.33$, $B = 5$, and $c = 0.02$. The pattern is similar to that in Fig. 2(f) and is used here to model the flow in the burner. Taking into account the turbulent character of the flow in the burner and interpreting the viscosity as some appropriate eddy viscosity, we disregard the no-slip condition and replace some streamlines by walls in Fig. 11(b). The streamlines $\psi=2$ shown by a bold curve in Fig. 11(b) serve as the
The flame front is expected to be axisymmetric and smooth and separates into inlet \( n \) enters through inlet \( C \), and hot outflow \( H \) leaves the tube through outlet \( OH \) near valve \( V \). Surface \( S \) separates \( C \) and \( H \), and on \( S \). (c) Meridional flow from the composite solution (10) giving streamlines with \( \psi = -50, -25, -5, 0, 5, 25, 50 \).

To model the flame front, we do not address here the heat and reaction-diffusion equations, but instead apply the following approach. If the contribution to heat and mass transfer from diffusion is small in comparison with that from advection, then one can neglect the thickness of the flame front and consider the front as a surface. The temperature and concentration are uniform and have the ignition values of the flow and concentration are uniform and have the ignition values of the flow.

Representing \( F = z - Z(r) \) and substituting in (12a) yields

\[
dZ/dr = \left[ \nu_c \left( \nu_c^2 + \nu_t^2 \right) - \nu_t \nu_c \right] / \left( \nu_c^2 - \nu_t^2 \right).
\]

The flame front is expected to be axisymmetric and smooth [see Fig. 11(a)]. In this case, \( dZ/dr = 0 \) at \( r = 0 \) and, since \( \nu_t = 0 \) at \( r = 0 \), it follows from (12b) that \( \nu_c = \nu_c(0, Z_0) \). Here \( Z_0 = Z(0) \) corresponds to an intersection point of the flame front with axis \( z \). It is convenient to start the integration of (12b) this point. For physical reasons, this point must be located in the interval of the \( z \) axis downstream of the origin, \( r = z = 0 \), but upstream of \( RZ \) in Fig. 11(b).

The flame propagates upstream (to the region where there is unburned fuel), and the advection velocity must be positive to stagnate the front. A particular value of \( \nu_c \) depends on combustion components. For example, \( \nu_c = 0.3 \) m/s for combustion of pure methane in air corresponds to the marginal value of self-ignition. In our problem, a given \( \nu_c \) specifies \( Z_0 \), and vice versa. Since the \( \nu_c \) value is small in comparison with the maximal velocity in the burner, the flame front is located slightly upstream of the bubble. The thick curve modeling the flame front in Fig. 11(b) starts at \( z = Z_0 = 15 \), while the bubble starts at \( z = 16.5 \).

To summarize, the analytic solutions (2), (10), and (11) are clearly useful in modeling rather complex practical flows. Certainly, each application considered above (along with many others) requires more thorough treatment. Our aim in Sec. VI–VII has been to display samples of possible applications; their detailed considerations are subjects for additional studies. In the following discussion (Sec. VIII), we try to focus on some common features from the solutions obtained and their applications.

**VIII. FORMATION OF VORTEX FILAMENTS**

Slender vortex filaments, observed in nearly all transitional and turbulent flows, play a significant role in vortex dynamics and transfer processes. The filaments are often viewed as the “sinews and muscles” of turbulent flows. For example, cosmic jets involve large-scale motion in the interstellar and intergalactic medium in the form of ‘‘vortex...’’
filaments’’ (i.e., slender regions of high vorticity). Another well-known example is the wing tip vortex that develops behind aircraft.

Based on our solutions and examples of applications, we discuss here a general mechanism by which vortex filaments can form, driven by self-focusing of swirling flow (i.e., increases in the axial momentum and swirl velocity near the axis) triggered by some external factor. Above a delta wing of an aircraft, for instance, the external factors include the vortex sheets that separate from the wing leading edge and the cross-flow, which stimulates rolling up of the sheets into vortices with axial flow. The upper part of Fig. 12 (extracted from flow visualization) shows streamlines forming the core of a delta-wing vortex (the white strips). Measurements reveal that the core is an intense swirling jet with a sharp peak in the longitudinal velocity. This jet results from focusing of the longitudinal momentum and swirl in the roll up process.

The lower part of Fig. 12 is a schematic of this process (streamlines) drawn with the help of our solution (2) at \( a = -0.5 \), \( b = 0.25 \), \( c = 0 \), \( Re = -4 \), and \( S = -30 \). The curves differ only with the initial \( r \) and \( z \) values and model streamlines separating from the leading edge. The curves collapse onto the axis, which models the vortex core.

The mechanism of vortex core formation is as follows. The roll up of the separation surface leads to the generation of swirl (as shown by the streamlines in the lower part of Fig. 12). This swirl induces a pressure drop near the axis, thus attracting other streamlines to the axis; this further focuses the swirl, therefore, further decreasing pressure in the vortex core. This positive feedback leads to strong accumulation of the axial vorticity and momentum, i.e., to vortex filament formation.

In the present solution, the convergence of swirling flow is modeled by a sink. The potential-vortex component of the solution corresponds to the conservation of angular momentum during the streamline convergence. The singularity of the axial velocity (see term \( W_r(r/r_0)^{Re} \) in (2c) at \( Re < 0 \)) reflects a strong concentration of the axial momentum near the axis by the converging flow. The significant accumulation of kinetic energy in a near-axis region is modeled by the fact that all velocity components and pressure are infinite on the axis, according to (2) and (5). As noted above, the solution (2)–(5) is not valid at the filament center, where regularization (or ‘‘cut off’’) of velocity and pressure occurs by viscous effects, modeled here by the composite solutions (10)–(11). The radius of the vortex core where the solution (2) must match an inner solution can serve as the length scale \( r_0 \) used in (1)–(5).

The viscous action causing vortex regularization in practical flows is physically obvious. Solution (2) helps us to illuminate the focusing of swirling flows; this effect is less clear (but more important for applications) than the regularization. The focusing is similar to the accumulation effect occurring in colliding swirl-free jets (as in cumulative artillery shells), where a bipolar jet appears due to convergence of a conical annular jet that self-collides at the cone apex.

Solution (2) describes such a focusing of axial momentum, but also an additional effect a strong pressure drop on the axis owing to swirl concentration. Composite solution (10) also captures the opposite effect: an abrupt broadening of the vortex core, i.e., vortex breakdown. Divergence of stream surfaces is caused here by the interaction of a swirling jet with an ambient flow (Sec. V). Thus, the generalized vortex-sink solution (2)–(5) provides insight into the vortex filament formation, and the composite solution (10) illuminates vortex breakdown.

**IX. CONCLUDING REMARKS**

We have derived a new analytical solution of the Navier–Stokes equations, representing a generalization of the planar vortex sink, by adding an axial flow with radial shear. This generalized five-parameter flow on surfaces of revolution has been applied to model elements of whirlpools, cosmic jets, tornadoes, and recirculatory swirling motion in a sealed cylinder. We have extended this solution to the near-axis region via matching with swirling jet flows, thus obtaining composite solutions. The composite solutions have been applied for vortex breakdown, the Ranque–Hilsch effect, flows in menisci of electrospays, and in vortex burners. The modeling helps us to explain the important physical features of these vortex flows and serves a potentially quite useful new approach to parametric studies and optimization of practical swirling flows. We anticipate other applications such as cumulative jets in the collapsing bubbles of cavitation, and heat and mass transfer in swirling flows.

The present results can be generalized even further to include nonaxisymmetric flows. Recently, rather striking results have been obtained related to the instability and bifurcation of planar vortex-sink flows. Bifurcation of the vortex-sinks leads to Oseen’s solutions, corresponding to nonaxisymmetric flow patterns with a few spiral branches. To generalize Oseen’s solution via the addition of an axial flow will require a laborious analysis, including both numerical simulations and asymptotic solutions. This is certainly an important issue for further study, but beyond the scope of the present paper, in which we focus on the generalized axisymmetric solution.
This research was founded by the Air Force Office of Scientific Research Grant No. F49620-95-1-0302. The authors are grateful to Wade Schoppa for a review of the manuscript.


F. A. Williams, Combustion Theory, 2nd ed. (Addison-Wesley, Reading, MA 1984), p. 44.


