# Onset of Convection near a Point Source of Heat and Gravity

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A paradoxical symmetry breaking—development of a unidirectional flow via thermal instability—is predicted by an analytical solution for compressible convection near a point source of both heat and gravity. Such a flow can propel a cosmic hot body (e.g., a protostar) in a molecular cloud. This convection emerges at the critical Rayleigh number $\mathrm{Ra} = l(l+1)[2l(l+1)+1+3\mu_V/2]\gamma/(\gamma-1)$, where $l$ is the number of flow cells, $\mu_V$ is the second-to-first viscosity ratio, and $\gamma$ is the specific heat ratio.

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1. *Introduction.*—This study is motivated by collimated outflows observed in cosmic space. These outflows emerge from young stars, double stars, dying stars, and galaxy cores (whose masses range from $10^{-2}$ to $10^9$ times the mass of the Sun), have lengths from $10^{-1}$ to $10^6$ times the distance from the Sun to the nearest star, and have velocities ranging from a few km/s to relativistic [1–3]. The fact that the outflows occur in a variety of objects and over a large range of scales—and therefore seem to be a generic phenomenon—inspired us to formulate a simple gas-dynamics model. Differing from astrophysical flows in many aspects, our model incorporates the following important common features: (i) gravity, (ii) density gradients, and (iii) energy flux from a central body. These features alone appear sufficient for the development of large-scale outflows.

Here we focus on the outflow formation via thermal instability. As a diffuse molecular or/and dust cloud collapses under its self-gravitational attraction, a massive body (e.g., a protostar) emerges which constitutes the placental material out of which a star or galaxy core forms. We consider this process as quasisteady and causing a *gradual* increase in the gravity force and in the energy flux from the body to the ambient. The effects of energy flux and gravity are characterized here by a single dimensionless quantity—the Rayleigh number Ra (buoyant/viscous force ratio). As the body becomes more massive, Ra increases. The equilibrium state of rest exists, where the gravity force is balanced by the radial gradient of pressure. For small Ra, this rest state is stable, but as Ra exceeds a critical value $\mathrm{Ra}_{\mathrm{cr}}$, a flow emerges via thermal instability.

To simplify analysis, we consider gravity due to the body only and neglect that due to the cloud matter. Next, we use a far-field approximation where the distance from the central body $r$ is much larger than the radius of the body (i.e., a point-source model). At such large distances, we consider a cloud (filled with dust particles) optically thick and use the Rosseland (diffusive) approximation for radiative heat transfer [4]. Finally, we treat the cloud material as a perfect gas whose viscosity is due to background turbulence and magnetic field [1]. Thus, we come to the simple problem of onset of convection in a perfect gas near a point source of both heat and gravity.

The problem is also of theoretical interest: the analytical solution obtained is the first for thermal convection in a compressible fluid. In relation to the classical Rayleigh solution [5] (for a horizontal layer of an incompressible fluid), our solution describes the opposite limiting case: the Rayleigh solution is valid for a narrow spherical shell [6,7], while the outer-to-inner radius ratio is infinite in our case (relevant for cosmic outflows). The *compressible* convection problem here avoids the Boussinesq approximation used in our prior study [8] (of collimated buoyant, conical jets in an incompressible fluid) and addresses flows of Keplerian ($\mathbf{v} \sim r^{-1/2}$, typical of cosmic flows) and not of conical ($\mathbf{v} \sim r^{-1}$) similarity; $\mathbf{v}$ is the velocity vector.

2. *Similarity family.*—Consider steady flows of a compressible fluid governed by

$$\nabla \cdot (\rho \mathbf{v}) = 0, \tag{1a}$$

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mu\Delta\mathbf{v} - \nabla p + \mu(\mu_V + 1/3)\nabla(\nabla\cdot\mathbf{v}) + \rho\mathbf{g}, \tag{1b}$$

$$\rho c_p(\mathbf{v}\cdot\nabla)T = \kappa\Delta T + \Phi + (\mathbf{v}\cdot\nabla)p, \tag{1c}$$

$$p = R\rho T, \tag{1d}$$

where $\rho$ is density, $p$ is pressure, $T$ is temperature, $\mathbf{g}$ is the acceleration due to gravity, $\Phi$ is the viscous dissipation, $\mu$ is viscosity, $\mu_V$ is the ratio of the bulk (second) viscosity [9] to $\mu$, $\kappa$ is thermal conductivity, $R = c_p - c_v$ is the gas constant, $c_p$ and $c_v$ are the specific heats at constant pressure and volume, respectively. We assume that $\mu$, $\kappa$, $c_p$, and $c_v$ are constant.

At $\mathbf{g} = 0$, the system (1) permits the similarity solutions,

$$\rho = \rho_R \underline{\rho}(\theta)\underline{r}^{\delta}, \qquad p = p_R \underline{p}(\theta)\underline{r}^{-\delta-2},$$
$$T = T_R \underline{T}(\theta)\underline{r}^{-2\delta-2}, \tag{2a}$$

$$\mathbf{v} = v_R \underline{\mathbf{v}}(\theta)\underline{r}^{-\delta-1}, \qquad \underline{\mathbf{v}} = \{u, v/\sin\theta, \Gamma/\sin\theta\}, \tag{2b}$$

where $(r, \theta, \phi)$ are spherical coordinates, $r$ is the distance from the coordinate origin, $\theta$ and $\phi$ are polar and

  

azimuthal angles, $\underline{r} = r/r_R$, $r_R$ is a length scale (e.g., the body radius), and $\delta$ is an arbitrary real number. We use $\rho_R$, $\mu$, $R$, and $r_R$ to scale velocity, temperature, and pressure:

$$v_R = \mu/(\rho_R r_R), T_R = \mu^2/(R\rho_R^2 r_R^2), \quad \text{and}$$
$$p_R = R\rho_R T_R = \mu^2/(\rho_R r_R^2), \tag{3}$$

where the index $R$ denotes reference values (e.g., at $r = r_R$).

At $\mathbf{g} \neq 0$, similarity solutions exist if $\mathbf{g}$ is proportional to $r^{-2\delta-3}$:

$$\mathbf{g} = -\mathbf{e_r} g_R \underline{r}^{-2\delta-3}, \tag{4}$$

where $g_R$ is the gravity acceleration at $r = r_R$, and $\mathbf{e_r}$ indicates the outward radial direction.

Use of (2)–(4) reduces (1) to the system of ordinary differential equations (dropping underline):

$$\rho' = \rho(u - v')/v, \tag{5a}$$

$$(1 - x^2)u'' = 2xu' - a_u u - b_u v' - (\delta + 2)p - \rho\{(1 + \delta)u^2 + vu' + (v^2 + \Gamma^2)/(1 - x^2)\} + \mathrm{G}r\rho, \tag{5b}$$

$$\mu^* v'' = a_v u' - p' - \{b_v v + \rho[\delta uv + vv' + x(v^2 + \Gamma^2)/(1 - x^2)]\}/(1 - x^2), \tag{5c}$$

$$(1 - x^2)\Gamma'' = -b_v\Gamma - \rho\{\delta u\Gamma + v\Gamma'\}, \tag{5d}$$

$$(1 - x^2)T'' = 2xT' - a_T T + \Pr\{(1 - 1/\gamma)[(\delta + 2)up + vp' - \Phi] - \rho[2(\delta + 1)uT + vT']\}, \tag{5e}$$

$$p = \rho T, \tag{5f}$$

where $x = \cos\theta$, the prime denotes differentiation with respect to $x$, $\mathrm{Gr} \equiv \rho_R^2 g_R r_R^3/\mu^2$ is the Grashof number, $\Pr \equiv \mu c_p/\kappa$ is the Prandtl number, and $\gamma \equiv c_p/c_v$ is the specific heat ratio. Note that $\mu^* = (4/3 + \mu_V)$, $a_u = \mu^*(\delta + 2)(\delta - 1)$, $b_u = (8 + \delta)/3 + \mu_V(\delta + 2)$, $a_v = 2 + (\mu_V + 1/3)(1 - \delta)$, $b_v = \delta(1 + \delta)$, and $a_T = 2(1 + \delta)(1 + 2\delta)$.

Use of (5f) and $p' = \rho[T' + T(u - v')/v]$ [resulting from differentiating (5f) and substituting $\rho'$ from (5a)] in (5b)–(5e) excludes pressure and makes the system resolved with respect to the highest derivatives. Since the system (5) is of the 9th order, we need nine conditions. The regularity requirements, for the velocity and stresses to be bounded on the axis, $x = \pm 1$, are

$$v = 0, \tag{6a}$$

$$\Gamma = 0, \tag{6b}$$

$$2xu' - a_u u - b_u v' - (\delta + 2)p - \rho(1 + \delta)u^2 + \mathrm{G}r\rho = 0, \tag{6c}$$

$$2xT' - a_T T + \Pr\{(1 - 1/\gamma)[(\delta + 2)up - \Phi] - 2\rho(\delta + 1)uT\} = 0. \tag{6d}$$

They follow from (2b) resulting in (6a) and (6b), and from (5b) and (5e) which are reduced to (6c) and (6d) by putting $x^2 = 1$. Since (6a)–(6d) must be satisfied at both $x = -1$ and $x = 1$, we have eight boundary conditions. The 9th condition is a normalization of $\rho$, since there is a freedom in choosing $\rho_R$. We put $\rho_R$ to be the averaged density at $r = 1$. This yields the integral condition,

$$\int_{-1}^{1} \rho\, dx = 2. \tag{6e}$$

Now the problem (5)–(6) is mathematically closed. An important feature is that the conditions (6) do not guarantee that the density is bounded on the axis, because the right-hand side of (5a) can be singular according to (6a). Assuming that the velocity permits Taylor expansion near $x = 1$,

$$v(x) = v'(1)(x - 1) + O[(x - 1)^2],$$
$$u(x) = u(1) + O(x - 1),$$

we find from (5a) that the leading term in the power expansion for $\rho(x)$ is $C(1 - x)^a$, where $a = u(1)/v'(1) - 1$ and $C$ is a constant. For $a \neq 0$, the density is either infinite ($a < 0$) or zero ($a > 0$) at $x = 1$; both the cases are unphysical. Thus, to obtain a regular solution, we need to satisfy the additional condition, $a = 0$ [i.e., $u(1) = v'(1)$], by choosing appropriate values of the parameters involved (say, Pr or $\mu_V$).

This difficulty does not occur for an incompressible fluid [8] where (5f) is omitted and $\delta = 0$ in (2). In that case, (5a) becomes $v' = u$; then use of $v' = u$ and $v'' = u'$ in (5c) reduces (5) to an 8th order system, and the conditions (6a)–(6d) close the problem. We reiterate that the compressible case does not reduce to the incompressible one by putting $\rho = \text{const}$ only; in addition, (5f) must be omitted. The fact that the incompressible problem cannot be treated as a specific case of the compressible problem has significant consequences. In particular, the solution by Landau [9] for a jet generated by a point source of momentum cannot be generalized for a compressible fluid. Fortunately, in the linear problem of onset of convection, the additional condition ($a = 0$) is satisfied automatically as we show below.

3. *Keplerian convection.*—For gravity due to a point source, $\mathbf{g}$ is proportional to $r^{-2}$, i.e., $\delta = -1/2$ [see (4)], and, therefore, $|\mathbf{v}| \sim (|\mathbf{g}|r)^{1/2} \sim r^{-1/2}$ (Keplerian similarity). Now, (2a) yields that $T \sim 1/r$; this corresponds to a point source of heat.

Thus, similarity solutions (2) with $\delta = -1/2$ describe flows near a point source of both heat and gravity,

$$\mathbf{g} = -\mathbf{e_r} g_R r^{-2}, \tag{7a}$$

$$T = T_R r^{-1}, \tag{7b}$$

where $T_R$ (temperature at $r = 1$) is now a control parameter, characterizing the total heat flux from the point source.

First, we consider the equilibrium state of rest, i.e., $\mathbf{v} = 0$. In this state, $p$, $\rho$, and $T$ depend on $r$ only. Using (1d) and (7b) we transform the reduced equation (1b), $dp/dr = -\rho g_R r^{-2}$, into

$$dp/dr = -p(RT_R)^{-1} g_R r_R r^{-1},$$

and, substituting for $p$ from (2a) with $\delta = -1/2$, we obtain the equilibrium condition,

$$g_R r_R = 3RT_R/2. \tag{8}$$

This relation determines the gravity force that balances the radial gradient of pressure induced by the heat source.

Thus, in the equilibrium state of rest, (2a) reduces to

$$T = T_R T_e r^{-1}, \qquad \rho = \rho_R \rho_e r^{-1/2}, \qquad p = p_R p_e r^{-3/2}, \tag{9}$$

where subscript $e$ denotes the dimensionless equilibrium values. Now (6a)–(6e) yield that

$$\rho_e = 1, p_e = T_e = 2\mathrm{Gr}/3. \tag{10}$$

4. *Infinitesimal disturbances of the equilibrium state.*—Now, we consider onset of thermal convection. To this end we examine the disturbed quantities:

$$\mathbf{v} = \mathbf{v}_d, \qquad \rho = \rho_R(1 + \rho_d) r^{-1/2}, \qquad p = p_R(p_e + p_d) r^{-3/2}, \qquad T = T_R(T_e + T_d) r^{-1}, \tag{11}$$

where subscript $d$ denotes disturbances.

Substituting (11) into (1) and neglecting nonlinear terms with respect to disturbances, we obtain the system governing infinitesimal disturbances,

$$\nabla \cdot (\mathbf{v} r^{-1}) = 0, \tag{12a}$$

$$\nabla(r^{-3/2} p_d) = \Delta(r^{-1/2}\mathbf{v}) + (\mu_V + 1/3)\nabla(\nabla \cdot (r^{-1/2}\mathbf{v})) - r^{-5/2} 3/2(p_d - T_d)\mathbf{e_r}, \tag{12b}$$

$$r^3 \Delta(T_d r^{-1}) + u\mathrm{Ra}(1/\gamma - 1/3) = 0, \tag{12c}$$

$$\rho_d = 3(p_d - T_d)/(2\mathrm{Gr}), \tag{12d}$$

where subscript $d$ is omitted for velocity, and $\mathrm{Ra} \equiv \mathrm{PrGr}$ is the Rayleigh number. We see that the equation (12d) for $\rho_d$ is decoupled from (12a)–(12c); this resolves the regularity problem (discussed in §2).

5. *Critical Rayleigh numbers for convection onset.*—For axisymmetric swirl-free disturbances, using (2b) with $\Gamma \equiv 0$, we reduce the system (12), to

$$v' = u, \tag{13}$$

$$L^2 T_d + \mathrm{Ra}(1/\gamma - 1/3)u = 0, \tag{14}$$

$$L^2 u = (1 + 3\mu_V/2)u/2 + 3/2T_d = 0, \tag{15}$$

where $L^2 f \equiv (1 - x^2)f'' - 2xf'$. Applying the $L^2$ operator for (15) and substituting for $L^2 T_d$ from (14), we deduce a single equation governing the radial velocity $u$,

$$L^2[L^2 u - (1 + 3\mu_v/2)u/2] = u\mathrm{Ra}(3/\gamma - 1)/2. \tag{16}$$

Using boundary conditions (6), we find that solutions of (16) are the Legendre polynomials, $u = P_l(x)$, $l = 1, 2, \ldots$. The fact that the Legendre polynomials satisfy the equation $L^2 u = -l(l + 1)u$ allows us to obtain from (16) an analytical expression for critical values of Ra:

$$\mathrm{Ra}_{\mathrm{cr}} = l(l + 1)[2l(l + 1) + 1 + 3\mu_V/2]\gamma/(3 - \gamma), \qquad l = 1, 2, \ldots. \tag{17}$$

Now, (15) yields that

$$T_d = P_l(x)[2l(l + 1) + 3\mu_V/2]/3. \tag{18}$$

Note that these neutral solutions have alternating symmetry with respect to the equatorial plane: $u$ and $T_d$ are symmetric functions of $x$ for even $l$ and antisymmetric for odd $l$. Since we study the problem where there is no mass flux from the source, the $l = 0$ case is excluded.

Integrating (13) under the condition $v(1) = 0$ gives $v(x)$. Note that the condition $v(-1) = 0$ is automatically satisfied due to the symmetry with respect to $x = 0$ (see the examples in the next section). The $\theta$ projection of (12b) follows from (5c) after neglecting the nonlinear terms and putting $\delta = -1/2$ and $u' = v''$:

$$12p_d' = (14 + 6\mu_V)v'' + 3v/(1 - x^2), \tag{19}$$

whose integration gives the pressure disturbance $p_d$. Finally, the density disturbance follows from (12d). The condition, $\int_{-1}^{1} \rho_d dx = 0$, gives a value of integration constant for (19).

6. *Neutral modes for a few small values of $Ra_{\mathrm{cr}}$.*

*6.1 Unidirectional flow.*—At $l = 1$ (one-cell convection), the critical Rayleigh number is minimal, $\mathrm{Ra}_{\mathrm{cr}} = 2(5 + 3\mu_V/2)\gamma/(3 - \gamma)$, and the neutral mode is

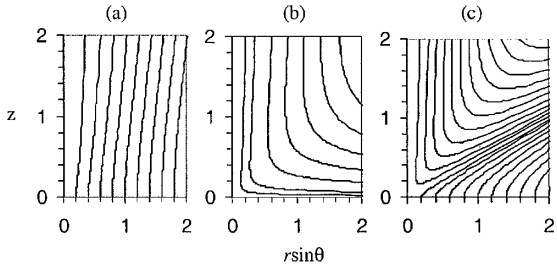


FIG. 1. Convection patterns for $l = 1$ (a), $l = 2$ (b), and $l = 3$ (c).

$$u = x, v = (x^2 - 1)/2, T_d = (5/3 + \mu_V/2)x\,,$$
$$p_d = (25/24 + \mu_V/2)x, \rho_d = -15x\mathrm{Pr}/(16\mathrm{Ra}_{\mathrm{cr}})\,. \tag{20}$$

We have normalized this linear solution by the condition $u(1) = 1$.

*6.2 Bipolar outflow.*—The presence of the accretion disk makes the radial velocity symmetric with respect to the $x = 0$ plane. As (20) does not satisfy this requirement, we consider the next case, $l = 2$ (two-cell convection), where $\mathrm{Ra}_{\mathrm{cr}} = 6(13 + 3\mu_V/2)\gamma/(3 - \gamma)$, and the neutral mode is

$$u = (3x^2 - 1)/2, v = (x^3 - x)/2\,,$$
$$T_d = (13/3 + \mu_V/2)u\,, \tag{21}$$
$$p_d = (27/8 + \mu_V/2)u, \rho_d = -23u\mathrm{Pr}/(16\mathrm{Ra}_{\mathrm{cr}}).$$

*6.3. Flow patterns.*—Convection cells are separated by surfaces $\Psi = 0$; $\Psi$ is the Stokes stream function. According to the relations, $v_r = (\rho r^2 \sin\theta)^{-1}\partial\Psi/\partial\theta$ and $v_\theta = -(\rho r \sin\theta)^{-1}\partial\Psi/\partial r$, we get that $\Psi = -v(x)r^{3/2}2\mu r_{\mathrm{R}}/3$. Hence, $\Psi = 0$ where $v(x) = 0$, i.e., the cells are separated by conical surfaces, $\theta = \mathrm{const}$, and the number of cells equals $l$. Flow patterns of the neutral modes shown in Fig. 1 are symmetric with respect to both the abscissa and the ordinate, $z = r\cos\theta$.

7. *Concluding remarks.*—Our results provide an analytical solution for thermal convection in a perfect gas. No analytical solution was reported so far presumably because the problem is significantly complicated without the Boussinesq approximation (invalid for compressible convection). In particular, there is a difficulty in satisfying the regularity requirement for density disturbances (§2). This difficulty is resolved here for the linear problem: by excluding the density disturbances from the equations for velocity, pressure, and temperature (12a)–(12c) and then using (12d), density is rendered bounded. [Such simplification is not obvious in the nonlinear problem, and further studies are needed.]

Our solution predicts a striking symmetry-breaking effect: the development of a *unidirectional* buoyancy-driven flow from a spherically symmetric state. This flow exerts thrust on the central body (as in a rocket); this constitutes the main difference between the compressible and incompressible cases. In the incompressible case, a unidirectional flow cannot be driven by buoyancy [8]; in the compressible case, the unidirectional flow is driven by buoyancy, emerging via thermal instability at $\mathrm{Ra}_{\mathrm{cr}} \neq 0$ (§6.1). At first sight, the development of a unidirectional flow from a spherically symmetric state seems paradoxical. However, the underlying mechanism is due to compressibility. Flow deceleration typically increases the gas density. For example, in a steady pipe flow, where $\rho u = \mathrm{const}$, a decrease in velocity causes an increase in density; this shifts the center of gas mass downstream. A similar shift occurs in our problem and provides thrust.

The thrust results from pressure distribution. According to the Boussinesq approximation (applied in the incompressible case), density disturbances (linearly) depend on temperature disturbances only. This produces no pressure disturbance in the linear solution and, since pressure remains spherically symmetric, no thrust. In contrast, in the compressible case, pressure disturbances occur and they are not spherically symmetric, being proportional to the radial velocity; see (20). The increased downstream pressure pushes the central body upstream in the $l = 1$ flow. Thus, our solution indicates that a massive cosmic body can be propelled by a unidirectional thermal-convection flow.

An important feature of this solution is Keplerian similarity that better fits cosmic flows, compared with conical similarity in the incompressible case [8]. We view the problem of onset of compressible convection near a point source of heat and gravity as a prototypical model that mimics the early stage of formation of large-scale outflows accompanying gravitational collapse near massive bodies in cosmic space.

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